An example of a stochastic equilibrium with incomplete markets

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Abstract. We prove existence and uniqueness of stochastic equilibria in a class of incomplete continuous-time financial environments where the market participants are exponential utility maximizers with heterogeneous risk-aversion coefficients and general Markovian random endowments. The incompleteness featured in our setting - the source of which can be thought of as a credit event or a catastrophe - is genuine in the sense that not only the prices, but also the family of replicable claims itself is determined as a part of the equilibrium. Consequently, equilibrium allocations are not necessarily Pareto optimal and the related representative-agent techniques cannot be used. Instead, we follow a novel route based on new stability results for a class of semilinear partial differential equations related to the Hamilton-Jacobi-Bellman equation for the agents' utility-maximization problems. This approach leads to a reformulation of the problem where the Banach fixed point theorem can be used not only to show existence and uniqueness, but also to provide a simple and efficient numerical procedure for its computation.

1. Introduction

Market incompleteness and equilibria. The central theme of this paper is a study of an equilibrium problem in an incomplete continuous-time stochastic setting. In contrast with the complete case where significant advances have been made in continuous time (see, e.g., [1, 7, 9, 10, 11, 16, 17, 18, 31] as well as Chapter 4. of [19]) the incomplete-market literature is lagging behind. Even among the few incomplete markets studied so far (see [2] or [14], for example) ideas related to market completeness, typically through the representative-agent approach, are used. To the best

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of our knowledge, the present paper is the only one in continuous time where a fully-incomplete market structure, in the sense that both the prices and the set of replicable claims (the marketed subspace) are determined as a part of the equilibrium, is analyzed and existence of equilibria is established.

The difficulty with incomplete markets is that Pareto optimality, which is commonly exploited to establish existence of equilibrium in a complete market, is not guaranteed anymore. Our approach rests upon the notion of stability of demand - a continuity property of the optimal response (optimal portfolio) when viewed as a function of market dynamics (the market-price-of-risk process). The leitmotif behind such an analysis is the following: if the aggregate demand can be shown to possess good continuity properties in an appropriate topological setting, a fixed-point-type theorem can be used to guarantee existence and (with some luck) uniqueness of an equilibrium market dynamics.

Stability in Hölder spaces. The technical bulk of the present manuscript is devoted to the stability of the optimal investment strategy under small functional perturbations of the market-price-of-risk coefficient. Problems similar to ours have attracted some attention recently - see, for example, [4, 5, 15, 20, 23, 24]. There are at least three - at first glance very different - reasons why such problems are important.

First, from the statistical and financial point of view, it is important to understand how misspecification or misestimation of the market dynamics coefficients affects the optimal trading strategies of agents who take the coefficient estimates at face value. Equivalently, one can wonder what kinds of statistical procedures for the estimation of those coefficients yield the most stability in implementation. For a deeper discussion of this point, we refer the reader to [24].

Second, in agreement with the classical methodology of the theory of partial differential equations, and applied mathematics in general, the following three aspects of every new problem are typically studied: existence, uniqueness and sensitivity of the solution with respect to changes of the problem's input parameters. These criteria are generally known as *Hadamard's well-posedness requirements* (see [13]). We view model specification as one of the most important input data in the utility-maximization problem, and understand the stability with respect to it as one of Hadamard's requirements.

The last reason - by far the most important one for the present paper - is to shed more light on the intimate relationship between the notions of stability and that of competitive equilibrium in incomplete markets. Indeed, rationality in the presence of individual preferences can be viewed as the "law of motion" of financial agents. In aggregate over several investors, the optimal trading strategies can be interpreted as the agregate demand for the assets in question. This aggregate demand, via the principles of market clearing, ultimately determines the shape of the price-dynamics of the financial assets.

Unlike other stability-related results mentioned above, the present paper approaches the problem from a different point of view. While all the previous strategies involved probabilistic and convex-analytic techniques, we tackle the problem from a pure PDE perspective. In this way we are able to draw much stronger and more precise conclusions about the behavior of the function which maps the market-price-of-risk process into the optimal portfolio. Our main stability result is that this map is *locally Lipschitz continuous* when the inputs and the output are placed in

anisotropic Hölder spaces on $[0, T] \times \mathbb{R}$, and we give explicit estimates on the Lipschitz constant (see Theorem 2.6). The price we pay for the PDE approach is not as dire as one may think when placed in the equilibrium setting - the necessary Markovian assumption only restricts the form of the agents' random endowment to functions of state variables. The Markovian structure of the resulting equilibrium asset dynamics is, on the other hand, not a restriction. It can be viewed as a strenghtened version of an equilibrium existence result: not only do we show that an equilibrium exists, we also show that a *Markovian* equilibrium exists.

In order to illustrate our techniques with the minimal amount of distraction, we choose what can be termed as the simplest nontrivial incomplete market model. Indeed, our market consists of a single risky asset (and a unit riskless asset) whose dynamics depend on a single Brownian motion and a one-jump Poisson process. All the incompleteness in the market comes from this uninsurable and unpredictable jump. Every agent is an exponential utility maximizer who also receives a random endowment at the end of the time horizon and this random endowment is allowed to depend on all sources of uncertainty in the market, including the indicator of the unpredictable jump. As a form of a normalization, we assume that the volatility of the risky asset is constant, but the drift can depend on time and the current value of both state processes - the Brownian motion and the indicator of the unpredictable jump. It is clear that our setting can be generalized in many different directions and that our estimates are far from optimal. Our goal was to provide a proof of concept for the powerful PDE techniques outlined in the body of the paper. We strive to keep the presentation as simple as possible and as accessible as possible to a reader who is not a specialist in PDE. For that reason, we use only elementary techniques and results in the Schauder-type theory of semilinear parabolic PDE as outlined, for example, in [21]. In the same spirit, we do not pursue any connections with BSDE.

Existence and uniquencess of the equilibrium price dynamics. Once the local Lipschitz continuity of the agents' demand functions is established, we turn to the market-clearing conditions and show that they can be rephrased in terms of a simple fixed-point problem for a continuous map on an anisotropic Hölder space. Surprisingly, under a "smallness" condition, a restriction of this map becomes a *contraction* and therefore, both existence and uniqueness of equilibrium can be guaranteed. Furthermore, an efficient and easy-to-implement numerical technique - based on iteration - emerges naturally. Results of this type are very rare in equilibrium theory; equilibria are typically not unique and their multiplicity is often one of the major objections to the use of equilibrium modelling in practice. Also, the computational methods (even in the fully-finite-dimensional case) are often quite involved and based on inefficient and relatively hard-to-implement procedures based on Sperner's lemma and the related Scarf's algorithm (see [28] and [29]).

The structure of the paper. Section 2. desribes the model and provides statements and some proofs of our main results. Section 3. is devoted to the proof of the central stability results, while Appendix A contains the pertinent information about Hölder spaces.

2. The financial environment and market clearing

We define our financial environment by specifying the structure of the three main ingredients: the information structure, agents' preferences and the completeness constraints.

2.1. **The information structure.** Let $T > 0$ be a real number which we interpret as the time horizon, and let $\{B_t\}_{t\in[0,T]}$ be a standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that, additionally, the same probability space accommodates an independent, exponentially distributed random variable τ with parameter $\mu > 0$, where the corresponding counting process $\{N_t\}_{t\in[0,T]}$ is given by

$$N_t = \mathbf{1}_{\{\tau \le t\}},\ t \in [0, T].$$

Let $\{\mathcal{F}_t\}_{t\in[0,T]}$ be the right-continuous augmentation of the filtration $\{\mathcal{F}^B_t \vee \mathcal{F}^N_t\}_{t\in[0,T]}$, where $\mathcal{F}^B_t = \sigma(\{B_s\}_{s\le t})$ and $\mathcal{F}^N_t = \sigma(\{N_s\}_{s\le t})$ for $t \in [0, T]$.

A more detailed interpretation of the roles of the two filtrations will be given later. For now, let us just mention that the σ-algebra $\mathcal{F}_t$ models the total publicly-available information at time t and that τ models an event with a significant impact on the market. Our model does not deal with asymmetric-information situations - all the agents have access to the same information, and no information is hidden from any of them. As we shall explain below, it is the way in which the information trickles into the dynamics of traded assets that makes the market incomplete.

2.2. **Completeness constraints.** The final goal of our analysis is the determination of the form of an equilibrium asset-price process. Without exogenously-imposed constraints, there is nothing that will prevent the agents from "opening" as many markets as possible, and, eventually, building a market structure that will be able to replicate any uncertain pay-off. In other words, without constraints - and bar pathologies - all equilibrium markets are necessarily complete. The situation we are modelling, however, calls for a degree of incompleteness: we envision the situation in which the random variable τ marks an event with the property that the time-scale at which the market adjusts to τ is larger than the time-scale on which it reacts to the other information. Consequently, as we shall see, its nature will be such that no perfect insurance against its effects can be bought or sold.

More generally, market environments where a portion of the information flow affects prices faster than the rest are important examples of natural constraints that preclude completeness; we term them **fast-and-slow-information market environments**. In fact, many widely used incomplete market models (virtually all models where the incompleteness stems from more "sources of uncertainty" than available risky assets) can be viewed as equilibria in fast-and-slow-information market environments. The situation described in the present paper corresponds to, arguably, the simplest continuous-time fast-and-slow-information market environment. More complicated (and realistic) environments can be constructed and analyzed, but we opt for one which allows us to showcase our methods without unnecessary confusion.

The defining properties of τ lead to the following class of possible asset-price dynamics: let Λ be the family of all $\{\mathcal{F}_t\}_{t\in[0,T]}$-predictable processes $\{\lambda_t\}_{t\in[0,T]}$ such that $\int_0^T |\lambda_u|\, du < \infty$, a.s., and let $\mathcal{S} = \Big\{\{S^{(\lambda)}_t\}_{t\in[0,T]}\ :\ \lambda \in \Lambda\Big\}$ be the family of all Itô-processes $\{S^{(\lambda)}_t\}_{t\in[0,T]}$ with the dynamics given by

$$dS^{(\lambda)}_t = \lambda_t\, dt + dB_t,\ S^{(\lambda)}_0 = 0, \tag{2.1}$$

as λ ranges over Λ. Additionally, let Λ^{bd} be the subset of Λ consiting of all uniformly bounded λ. The corresponding set of price processes is denoted by $\mathcal{S}^{\mathrm{bd}}$. We always (implicitly) assume that a risk-free numéraire asset $\bar{S} \equiv 1$ accompanies each $S^{(\lambda)}$.

Remark 2.1.

(1) It is important to note that there is no loss of generality in assuming that the volatility coefficient is identically equal to 1 or that "arithmetic" dynamics for $S^{(\lambda)}$ are chosen instead the more common "geometric" one. Indeed, the only property of the asset-price dynamics important for the determination of the equilibrium is the set of replicable random variables it produces, and this depends on drift and volatility only through their quotient λ (the so-called market price of risk). In fact, our formulation of the equilibrium simply *cannot* distinguish between the two. This would be the case in the more complicated (and more realistic) model where the agents' random endowments depend on the asset-price itself, i.e., where the agents hold financial derivatives of the asset. Similarly, the postulated availability of a trivial numéraire asset $\bar{S}$ involves a minimal loss of generality. Indeed, the agents' utilities depend solely on the terminal wealth and do not exhibit any time-impatience characteristics. In fact, without introducing consumption into the model, it is impossible to disentangle the two solely on the basis of equilibrium analysis.

(2) The two terms on the right-hand side of (2.1) contribute to the fluctuations in $S^{(\lambda)}$ on different scales; the order of magnitude of the term $\lambda\, dt$ is dt while the order of (the absolute value of) the term dB_t is $\sqrt{dt}$. Equivalently, the first term takes on average $1/\sqrt{dt}$ times longer to produce the same (local) efect as the second one. In this sense, the fact that the new information regarding τ comes in only through the first, slower, term corresponds exactly to the requirement that the occurrence of τ be absorbed into the asset dynamics on a slower scale than the shocks produced by B.

(3) One can view (2.1) as a restriction on the set of stochastic processes that may serve as allowable market dynamics. For a different specification of the characteristics of the market environment (constraints on the information flow, number of assets, etc), we would get different families of processes. Therefore, one could give the following, abstract, definition: a **completeness constraint** is simply a family $\mathcal{S}$ of (possibly multi-dimensional) semimartingales.

(4) While no dividend structure has been mentioned so far, there is a way of incorporating dividends in the present model, and, perhaps, the best way to describe it is to compare it to a particular incomplete setting - the one with short-lived securities - in the language of [25] (see also how it leads to existence results in discrete time as in [25], Proposition 25.1, p. 255). One can think of the asset $S^{(\lambda)}$ as an aggregation of short-lived assets with inception t and maturity $t + dt$, each of which pays a dividend $S^{(\lambda)}_{t+dt}$ and costs $S^{(\lambda)}_t$. Equivalently, the agent enters into a bet a unit of which pays dB_t and costs $-\lambda_t\, dt$. In this way, we can think of dB_t as the local dividend and $-\lambda_t$ as its price. It should be noted that it is easy to incorporate any (reasonably behaved) martingale M_t of the form $M_t = \int_0^t \sigma_u\, dB_u$ as the divided process - one has to replace $S^{(\lambda)}$ by $\int_0^{\cdot} \sigma_u\, dS^{(\lambda)}_u$. Even if M is not a martingale, we can simply add its drift to $-\lambda$ and turn it into one. This procedure also allows us to give

a loose interpretation of the formation of volatility in the market. While the market price of risk λ is determined in the equilibrium from the agent's primitives and the structural properties of the dividends, the volatility is determined by their quantitative properties. Consequently, the unit-volatility assumption we impose on $S^{(\lambda)}$ can be reinterpreted as a simple normalization of the dividends.

2.3. **The agents and their preferences.** We assume there is a finite number $I \in \mathbb{N}$ of agents, all of whom actively participate in trading in all available assets. The preference structure of each one of them is determined by the following:

(I) the *utility functions:* each agent is an exponential-utility maximizer with

$$U^i(x) = -\exp(-\gamma^i x), x \in \mathbb{R},\ \gamma^i > 0.$$

(II) the *random endowment:* $\mathcal{E}^i = g^i(B_T, N_T)$, with bounded $g^i : \mathbb{R} \times \{0,1\} \to \mathbb{R}$, where further regularity conditions are to be specified.

As usual in the Alt-von Neumann-Morgenstern expected utility paradigm, agent i prefers the random variable $X_1 \in \mathcal{F}_T$ to a random variable $X_2 \in \mathcal{F}_T$ if and only if $\mathbb{E}[U^i(X_1+\mathcal{E}^i)] \geq \mathbb{E}[U^i(X_2+\mathcal{E}^i)]$, where we set $\mathbb{E}[Y] = -\infty$ whenever $\mathbb{E}[Y^+] = \mathbb{E}[Y^-] = +\infty$.

Remark 2.2.

(1) If one thinks about B and N as factors, (II) above states that the agents' random endowment depends on all factors driving the public information. In particular, the (conditional) distribution of the random endowment may change abruptly and considerably at time τ. One of the possible financial interpretations of the situation is that all agents hold (long or short) positions in assets whose pay-offs are affected by the occurrence of τ (default-sensitive derivatives, callable bonds, disaster-sensitive investments, etc.).

(2) The exponential nature of the utilities allows us to partially remove the assumption that all agents use the same (subjective) probability to compute the expected utility of a particular position. Indeed, using the identity

$$\mathbb{E}^{\mathbb{P}^i}[-\exp(-\gamma^i(X+\mathcal{E}^i))] = \mathbb{E}[-\exp(-\gamma^i(X+\tilde{\mathcal{E}}^i))],$$

where $\tilde{\mathcal{E}}^i = \mathcal{E}^i - \frac{1}{\gamma^i}\log(\frac{d\mathbb{P}^i}{d\mathbb{P}})$, we can easily "absorb" different subjective probabilities into the form of the random endowment. Care must be taken, though, to ensure that the appropriate integrability conditions are met. Also, it should be noted that such a change of measure can lead to loss of the Markovian structure of the ingredients.

Let us now focus on the pertinent case when the set of tradable assets consists of a single risky asset given by $S^{(\lambda)} \in \mathcal{S}^{\text{bd}}$, for some bounded market-price-of-risk process $\lambda \in \Lambda^{\text{bd}}$ and the trivial numéraire asset $\bar{S}$ (as described just below equation (2.1), above). Agent i chooses a dynamic self-financing portfolio process in the appropriate admissibility class (to be specified shortly) so as to maximize the expected utility of terminal wealth:

$$\mathbb{E}[U^i(\textstyle\int_0^T \pi_u\, dS_u^{(\lambda)} + \mathcal{E}^i)] \to \max. \tag{2.2}$$

Here, the value π_t of the one-dimensional process $\{\pi_t\}_{t\in[0,T]}$ denotes the number of shares of the risky asset in the portfolio. We do not explicitly mention the number of shares ρ_t of the riskless

asset, since, thanks to the self-financing condition, it necessarily satisfies

$$\rho_t = \int_0^t \pi_u \, dS_u^{(\lambda)} - \pi_t S_t^{(\lambda)}. \tag{2.3}$$

Our class of admissible portfolio processes aims to be just restrictive enough to rule out doubling strategies, yet large enough to contain the maximizer of the utility-maximization problem (2.2). We would like to emphasize that, due to the regularity of some of the ingredients, one does not need the sophistication typically encountered in general semimartingale models (see, e.g., the classes Θ_i, $i = 1, 2, 3, 4$ in [8] or the notion of permissibility in [26]). Instead, we sacrifice a small amount of generality for a large gain in simplicity by proceeding as follows: the admissibility class $\mathcal{A}$ consists of all $\{\mathcal{F}_t\}_{t\in[0,T]}$-predictable processes $\{\pi_t\}_{t\in[0,T]}$ such that $\mathbb{E}[\exp(\int_0^T (\frac{1}{2}+\varepsilon)\pi_u^2 \, du)] < \infty$, for some $\varepsilon > 0$. Note that, by Hölder's inequality, the integral $\int_0^T \pi_u \, dS_u^{(\lambda)}$ is well-defined for each $\pi \in \mathcal{A}$ and $\lambda \in \Lambda$.

Remark 2.3. As we shall see later, the optimization problem (2.2) admits a $dt \times d\mathbb{P}$-unique maximizer $\pi^{(\lambda),i}$ of (2.2) in the class $\mathcal{A}$ for each $\lambda \in \Lambda^{\mathrm{bd}}$ and each agent $i = 1, \dots, I$. The portfolio process $\{\pi_t^{(\lambda),i}\}_{t\in[0,T]}$ is interpreted as an *optimal response* of agent i to the market dynamics induced by λ; the mapping $\{\lambda_t\}_{t\in[0,T]} \mapsto \{\pi_t^{(\lambda),i}\}_{t\in[0,T]}$ plays a central role in our analysis.

2.4. **Market-clearing conditions and the main result.** Having introduced all the ingredients, we turn to our main problem. A central economic principle is that the prevailing market dynamics must have the following fundamental property: the demand and supply for each tradable asset must match, i.e., all markets must clear. A precise definition follows:

Definition 2.4. A process $S^{(\lambda)} \in \mathcal{S}$ (or, equivalently, its market-price-of-risk process $\lambda \in \Lambda$) is said to be an **equilibrium price dynamics (market price of risk)** if there exist processes $\{\pi_t^{(\lambda),i}\}_{t\in[0,T]} \in \mathcal{A}$, $i = 1, \dots, I$ such that

(1) **(rationality)** for all $\pi \in \mathcal{A}$ and all for all $i = 1, \dots, I$,
$$\mathbb{E}[U^i(\textstyle\int_0^T \pi_u^{(\lambda),i} \, dS_u^{(\lambda)} + \mathcal{E}^i)] \geq \mathbb{E}[U^i(\int_0^T \pi_u \, dS_u^{(\lambda)} + \mathcal{E}^i)], \text{ and}$$
(2) **(market clearing)** $\sum_{i=1}^I \pi_t^{(\lambda),i} = 0$, for all $t \in [0,T]$, a.s.

Remark 2.5. The reader will immediately realize that we only require market clearing for the risky asset $S^{(\lambda)}$. The self-financing condition (2.3) implies, however, that the market for the riskless asset $\bar{S}$ will clear, in that case, as well.

The main result of the present paper - which asserts the existence, uniqueness and efficient computability of equilibrium price dynamics - is summarized in Theorem 2.6 below (we direct the reader to Appendix A for details on Hölder spaces and the related notation):

Theorem 2.6 (Existence, Uniqueness and Computability of Equilibria)**.** *Consider the setup given in subsections 2.1, 2.2 and 2.3, and assume, additionally, that $g^i(\cdot, n) \in C^{2+\alpha}(\mathbb{R})$, for $i = 1, \dots, I$, $n = 0, 1$ and some $\alpha \in (0,1]$. Then there exists a constant $T_0 > 0$ (which may depend on $\alpha, \mu, \{\gamma^i\}_{i=1,\dots,I}$ and $\{|g^i|_{2+\alpha}\}_{i=1,\dots,I}$) such that for $T \leq T_0$,*

(1) *the equilibrium price dynamics* exist *and the corresponding market-price-of-risk process is* unique *in the class of all processes $\{\lambda_t\}_{t\in[0,T]}$ such that $\lambda_t = \lambda(t, B_t, N_{t-})$, $t \in [0,T]$, for some function $\lambda \in C^{1+\alpha}([0,T] \times \mathbb{R} \times \{0,1\})$.*

(2) *the function* λ *is* efficiently computable, *i.e., there exists a subset* $D \subseteq C^{1+\alpha}([0,T] \times \mathbb{R} \times \{0,1\})$ *with* $0 \in D$ *and a contraction* $\Pi : D \to D$ *such that* λ *is the fixed point of* Π.

Remark 2.7.

(1) The most restrictive feature of Theorem 2.6 is the "smallness" condition we place on the size T of the time horizon. It is not evident that it can be dealt with by "pasting" equilibria together as the upper bound on T depends on the primitives of the model. In particular, extending an equilibrium from T to $T' > T$ would entail using the value of the original equilibrium at time T as the terminal condition of the equilibrium on $[T, T']$. An iteration of this procedure may very well lead to a bounded sequence.

We do conjecture, however, that this assumption *can* be relaxed if one only wants to establish the existence of the equilibrium, but *cannot* be relaxed if the additional benefits of uniqueness and efficient computability are also desired. The resolution of those important questions is the content of our future work and seem to require quite different and more sophisticated techniques. Moreover, preliminary numerical experiments suggest that the for a large class of realistic parameter values, the smallness constraint does not seem to be binding and the iteration procedure converges.

(2) It is, perhaps, instructive to interpret the above uniqueness result in the light of part (4) of Remark 2.1. What is truly unique is the equilibrium price of the local gamble of size dW_t. It follows that each *replicable* derivative security C admits a unique equilibrium price process S. The difference with the complete models is that the class of replicable contingent claims is determined as a part of the equilibrium and admits no simple description a priori.

The proof of Theorem 2.6 is based on the following stability estimate which, as discussed in the introduction, is of interest in its own right. We devote the entire section 3 to its proof. Since it deals with a single-agent's optimization problem, we omit the agent index i from its statement. Also, in the interest of readability, we introduce some additional notation: for a function $u : [0,T] \times \mathbb{R} \times \{0,1\} \to \mathbb{R}$, we define the n**-difference** u_n as

$$u_n(t,x,n) = u(t,x,1) - u(t,x,n) = \begin{cases} u(t,x,1) - u(t,x,0), & n = 0, \\ 0, & n = 1. \end{cases}$$

For such functions, restrictions $u(\cdot,\cdot,0)$ and $u(\cdot,\cdot,1)$ are called the $n = 0$- and the $n = 1$-**slices** of u. We say that a measurable function $f : [0,T] \times \mathbb{R} \times \{0,1\} \to \mathbb{R}$ is a **Markov representative** of the process $\{Y_t\}_{t \in [0,T]}$ if

$$Y_t = f(t, B_t, N_{t-}), \text{for all } t \in [0,T], \text{ a.s.}$$

It is convenient to identify a process admitting a Markov representative with the Markov representative itself. In fact, we will do so from this point on with little or no explicit mention.

Theorem 2.8 (Stability of the Optimal Portfolio)**.** *For* $\lambda \in C^{\alpha}([0,T] \times \mathbb{R} \times \{0,1\})$ *and* $g \in C^{2+\alpha}(\mathbb{R} \times \{0,1\})$ *there exists a unique function* $u^{(\lambda)} \in C^{2+\alpha}([0,T] \times \mathbb{R} \times \{0,1\})$ *which solves the following Cauchy problem for a semilinear PDDE (partial diferential-difference equation):*

$$\begin{cases} 0 = u^{(\lambda)}_t + \frac{1}{2} u^{(\lambda)}_{xx} - \lambda u^{(\lambda)}_x + \frac{1}{2\gamma}\lambda^2 - \frac{\mu}{\gamma}(\exp(-\gamma u^{(\lambda)}_n) - 1), \ \textit{on } [0,T) \times \mathbb{R} \times \{0,1\}, \\ u^{(\lambda)}(T,\cdot,\cdot) = g. \end{cases} \tag{2.4}$$

The stochastic process $\{\pi^{(\lambda)}_t\}_{t\in[0,T]}$ *with the Markov representative* $\pi^{(\lambda)} = \frac{1}{\gamma}\lambda - u^{(\lambda)}_x$*, i.e., the process given by*

$$\pi^{(\lambda)}_t = \tfrac{1}{\gamma}\lambda(t,B_t,N_{t-}) - u^{(\lambda)}_x(t,B_t,N_{t-}),\ t\in[0,T], \tag{2.5}$$

is in $\mathcal{A}$ *and attains the maximum in the utility-maximization problem* (2.2).

The maps $\lambda\mapsto\pi^{(\lambda)}$, $\lambda\mapsto u^{(\lambda)}_x$*, from* $C^\alpha([0,T]\times\mathbb{R}\times\{0,1\})$ *to itself, are locally Lipschitz continuous. More precisely, there exists a constant* $C=C(\gamma,\mu,\alpha)>0$ *such that for each* $R>0$ *and* $\lambda_1,\lambda_2\in C^\alpha([0,T]\times\mathbb{R}\times\{0,1\})$ *with* $|\lambda_1|_\alpha,|\lambda_2|_\alpha\le R$ *we have*

$$|u^{(\lambda_1)}_x - u^{(\lambda_2)}_x|_\alpha \le L(R)|\lambda_1-\lambda_2|_\alpha,\ \text{ and } |\pi^{(\lambda_1)}-\pi^{(\lambda_2)}|_\alpha \le \Big(\tfrac{1}{\gamma}+L(R)\Big)|\lambda_1-\lambda_2|_\alpha, \tag{2.6}$$

where

$$L(R) = C\,T^{(1+\alpha)/(2+\alpha)} e^{e^{2+2\gamma|g|_0+TR^2+2\mu T}}\Big(|g|_{2+\alpha}+(1+T)(1+R^2)\Big)^{6+4\alpha}. \tag{2.7}$$

With Theorem 2.8 at our disposal, the proof of Theorem 2.6 becomes straightforward. We start with a characterization which follows directly from Definition 2.4 and equation (2.5) of Theorem 2.8. When the index i is added to $u^{(\lambda)}$ (or its derivatives), as in $u^{(\lambda),i}$ (or $u^{(\lambda),i}_x$, etc.), we are referring to the solution to (2.4) with the agent-dependent terminal condition $g=g^i$ and the risk-aversion parameter $\gamma=\gamma^i$.

Lemma 2.9. *A process* $\{\lambda_t\}_{t\in[0,T]}$ *with Markov representative* $\lambda\in C^\alpha([0,T]\times\mathbb{R}\times\{0,1\})$ *is an equilibrium market-price-of-risk if and only if it is a fixed point of the operator* $\Pi : C^\alpha([0,T]\times\mathbb{R}\times\{0,1\})\to C^\alpha([0,T]\times\mathbb{R}\times\{0,1\})$ *defined by*

$$\Pi(\lambda) = \bar\gamma\sum_{i=1}^I u^{(\lambda),i}_x,$$

where $\bar\gamma = \Big(\sum_{i=1}^I \frac{1}{\gamma^i}\Big)^{-1}$.

In the sequel, let $R_0 = \frac{2}{\bar\gamma}\sum_{i=1}^I |u^{(0),i}_x|_\alpha$ (where $u^{(0),i}_x$ corresponds to $\lambda\equiv 0$) be computed for $T=1$. Note that R_0 dominates the value of the same expression we get when we set $T<1$.

Lemma 2.10. *There exists a constant* $T_1 = T_1(\alpha,\mu,\{\gamma^i\}_{i=1,\dots,I},\{|g^i|_{2+\alpha}\}_{i=1,\dots,I})>0$*, such that if* $T\le T_1$, Π *maps the ball*

$$B_\alpha(R_0) = \{\lambda\in C^\alpha([0,T]\times\mathbb{R}\times\{0,1\}) : |\lambda|_\alpha\le R_0\},$$

of radius R_0 *in* $C^\alpha([0,T]\times\mathbb{R}\times\{0,1\})$ *into itself.*

Proof. The first inequality in (2.6) of Theorem 2.8 implies that there exists a non-decreasing function $F:[0,\infty)\to[0,\infty)$ such that if $T\le 1$ we have

$$|u^{(\lambda),i}_x|_\alpha \le |u^{(0),i}_x|_\alpha + T^{1/(2+\alpha)}F(|\lambda|_\alpha),\ \text{ and so } |\Pi(\lambda)|_\alpha \le \tfrac12 R_0 + \tfrac{1}{\bar\gamma} I\,T^{1/(2+\alpha)}F(|\lambda|_\alpha),$$

for $\lambda\in C^\alpha([0,T]\times\mathbb{R}\times\{0,1\})$ and all $i=1,\dots,I$. Note that F, being derived from $L(\cdot)$ of (2.7), depends on α, μ, γ^i, $|g^i|_{2+\alpha}$, $i=1,\dots,I$ but not on T or λ for $T\le 1$. Therefore, for $T\le T_1 = 1\wedge\Big(\frac{\bar\gamma R_0}{2IF(R_0)}\Big)^{2+\alpha}$, we have $|\Pi(\lambda)|_\alpha\le R_0$, whenever $|\lambda|_\alpha\le R_0$. $\square$

Lemma 2.11. *There exists a constant $T_0 = T_0(\alpha, \mu, \{\gamma^i\}_{i=1,\dots,I}, \{|g^i|_{2+\alpha}\}_{i=1,\dots,I}) > 0$ such that if $T \leq T_0$, the mapping Π is a contraction from $B_\alpha(R_0)$ into itself.*

Proof. By Lemma 2.10, the mapping Π maps $B_\alpha(R_0)$ into itself, as long as $T \leq T_1$. Therefore, we can use the Lipschitz estimate (2.6) of Theorem 2.8 and reasoning similar to the one in the proof of Lemma 2.10 to conclude that there exists $T_0 \leq T_1$ such that for $T \leq T_0$, we have

$$|\Pi(\lambda_1) - \Pi(\lambda_2)|_\alpha \leq \tfrac{1}{2}|\lambda_1 - \lambda_2|_\alpha,$$

for $\lambda_1, \lambda_2 \in B_\alpha(R_0)$. □

Proof of Theorem 2.6. The path to the proof is paved by Lemmas 2.9, 2.10 and 2.11. Indeed, we simply apply the Banach fixed point theorem to the mapping Π on the complete metric space $B_\alpha(R_0)$ and note that, by the first part of Theorem 2.8, $\Pi(\lambda) \in C^{1+\alpha}([0,T] \times \mathbb{R} \times \{0,1\})$, as soon as $\lambda \in C^\alpha([0,T] \times \mathbb{R} \times \{0,1\})$. □

3. Stability of the optimal portfolio: a proof of Theorem 2.8

The purpose of the present section is to prove Theorem 2.8. We break its statement into several smaller, more manageable results, and proceed to discuss them one by one.

3.1. **The HJB equation: existence and verification.** The first assertion of Theorem 2.8 is that a the PDDE in (2.4) admits a regular solution and that it can be used to construct a Markov representative for the optimal portfolio in the utility maximization problem (2.2). Similar characterizations are ubiquitous in stochastic control in the case of exponential utility, and are too numerous to list; we simply instruct the reader to consult [6] and references therein. We note that a PDDE is a special case of a partial integro-differential equation, but we choose not to use that term because of the simplicity of the integral component. Equivalently, it could have been stated as a system of one linear and one semilinear PDE.

Appendix A should be consulted for notation not explicitly introduced in the main body of the paper. However, for the reader's convenience, we state the following convention both here and in subsection A.2 of Appendix A:

Convention 1.

(1) *The variables μ, γ and $\alpha \in (0,1)$ are considered "global" and will not change throughout the paper. Any function of the global variables (and global variables only) if called a* **universal** *constant.*

(2) *The notation $a \preceq b$ means that there exists a universal constant $C > 0$ such that $a \leq Cb$. Such a constant may change from line to line.*

Our analysis starts with a simple linear existence result in the spirit of Schauder's theory:

Lemma 3.1. *For $h, a \in C^\alpha([0,T] \times \mathbb{R})$ and $g \in C^{2+\alpha}(\mathbb{R})$ the Cauchy problem*

$$\begin{cases} 0 = u_t + \frac{1}{2}u_{xx} + hu_x + a \ \text{ on } [0,T) \times \mathbb{R}, \\ u(T,\cdot) = g(\cdot) \end{cases} \tag{3.1}$$

admits a unique solution $u \in C^{2+\alpha}([0,T] \times \mathbb{R})$. Moreover, for all $\beta > 0$ we have

$$|u|_{(\beta)} \quad \leq \quad \tfrac{1}{\beta}|a|_{(\beta)} + |g|_0. \tag{3.2}$$

Proof. The existence of a unique solution to (3.1) in $C^{2+\alpha}([0,T]\times\mathbb{R})$ is well-known (see, for example [21], Theorem 9.2.3, p. 140). To get (3.2), we use the fact that if u is the unique solution to (3.1), then $\tilde{u}(t,x)=e^{-\beta(T-t)}u(t,x)$ solves

$$\begin{cases} 0=\tilde{u}_t+\tfrac{1}{2}\tilde{u}_{xx}-\beta\tilde{u}+h\tilde{u}_x+\tilde{a} \text{ on } [0,T)\times\mathbb{R}, \\ \tilde{u}(T,\cdot)=g(\cdot), \end{cases} \tag{3.3}$$

where $\tilde{a}(t,x)=e^{-\beta(T-t)}a(t,x)$. It remains to note that the constant function $w(t,x)=|g|_0+\frac{1}{\beta}|\tilde{a}|_0=|g|_0+\frac{1}{\beta}|a|_{(\beta)}$ is a subsolution, and that its negative $-w$ is a supersolution of (3.3); the maximum principle (see [21], p.105, Theorem 8.1.2) implies

$$-w(t,x)\le\tilde{u}(t,w)\le w(t,x),\qquad (t,x)\in[0,T]\times\mathbb{R}. \tag{3.4}$$

□

When a non-linear term is added, a similar result can be obtained with a bit more work. The argument is based in part on the following lemma:

Lemma 3.2. *Let $\{x_n\}_{n\in\mathbb{N}_0}$ be a sequence of nonnegative real numbers with the property that*

$$x_{n+1}\le A+Bx_n^\alpha, \tag{3.5}$$

for some constants $A,B\ge 0$ and $0<\alpha<1$. Then there exists a constant $g(\alpha)>1$, independent of A,B and $\{x_n\}_{n\in\mathbb{N}}$, such that

$$\limsup_n x_n\le g(\alpha)\max(A,B^{1/(1-\alpha)}). \tag{3.6}$$

In particular, the sequence $\{x_n\}_{n\in\mathbb{N}}$ is bounded.

Proof. If $B=0$, the inequality in (3.6) clearly holds. When $A=0$ and $B>0$, we have $x_n\le B^{1+\alpha+\alpha^2+\dots+\alpha^{n-1}}x_0^{\alpha^n}$, which implies (3.6) directly.

Focusing on the case $A,B>0$, we set $\tilde{B}=B/A^{1-\alpha}$, and let $G>1$ be the unique positive solution to $G=1+\tilde{B}G^\alpha$. The scaled sequence $y_n=x_n/(AG)$, $n\in\mathbb{N}$ satisfies

$$y_{n+1}=\tfrac{1}{AG}x_n\le 1/G+\tilde{B}G^{\alpha-1}(y_n)^\alpha=\kappa+(1-\kappa)y_n^\alpha, \tag{3.7}$$

where $\kappa=G^{-1}=1-\tilde{B}G^{\alpha-1}$. Therefore, if $y_n\le 1$, then $y_{n+k}\le 1$, for all $k\in\mathbb{N}$. On the other hand, suppose that $y_n>1$, for all $n\in\mathbb{N}_0$. Then, for $n\in\mathbb{N}_0$, we have $y_{n+1}\le y_n^\alpha$, and so, $y_n\le y_0^{\alpha^n}$. Consequently, $\limsup_{n\to\infty}y_n\le 1$, i.e., $\limsup_n x_n\le AG$.

It remains to show that AG is bounded from above by the expression on the right-hand side of (3.6). Let $g(\alpha)$ be the unique positive solution of $g(\alpha)=1+g(\alpha)^\alpha$, so that $g(\alpha)>1$. If $\tilde{B}\ge 1$, then

$$1+\tilde{B}(g(\alpha)\tilde{B}^{1/(1-\alpha)})^\alpha=1+(g(\alpha)-1)\tilde{B}^{1/(1-\alpha)}\le g(\alpha)\tilde{B}^{1/(1-\alpha)}.$$

Therefore, the monotonicity of the function $x\mapsto x-1-\tilde{B}x^\alpha$ implies that $G\le g(\alpha)\tilde{B}^{1/(1-\alpha)}$. When $\tilde{B}\le 1$, we have $1+\tilde{B}g(\alpha)^\alpha\le g(\alpha)$, so $G\le g(\alpha)$. Therefore,

$$AG\le Ag(\alpha)\max(1,\tilde{B}^{1/(1-\alpha)})=g(\alpha)\max(A,B^{1/(1-\alpha)}).$$

□

The existence of a C^2 solution to semilinear equations of the type treated in Proposition 3.3 is well-known (see, for example, [3]). However, the $C^{2+\alpha}$-regularity and the $|\cdot|_0$ and $[\cdot]_{2+\alpha}$ estimates need additional work in this proof. The reader will note that we still go through the steps of the existence argument; the reason is that the Hölder regularity is established through the $|\cdot|_0$-convergent sequence which is originally constructed to show the existence of a C^2 solution.

Proposition 3.3. *Consider a semilinear Cauchy problem of the form*

$$\begin{cases} 0 = u_t + \frac{1}{2} u_{xx} + h u_x + a - b e^{\gamma u} \text{ on } [0,T) \times \mathbb{R}, \\ u(T,\cdot) = g, \end{cases} \tag{3.8}$$

where $g \in C^{2+\alpha}(\mathbb{R})$, $h \in C^{\alpha}([0,T]\times\mathbb{R})$, $a, b \in C^{\alpha}([0,T]\times\mathbb{R})$, $\gamma > 0$ and $a(t,x) \geq 0$, $b(t,x) \geq 0$, for all $(t,x) \in [0,T]\times\mathbb{R}$. Then (3.8) admits a unique solution in $C^{2+\alpha}([0,T]\times\mathbb{R})$. Furthermore, the following bounds hold for all $(t,x) \in [0,T]\times\mathbb{R}$:

$$-\tfrac{1}{\gamma} \log(e^{\gamma |g|_0} + \gamma T |b|_0) \leq u(t,x) \leq |g|_0 + T|a|_0, \tag{3.9}$$

$$[u]_{2+\alpha} \preceq [g]_{2+\alpha} + [a]_{\alpha} + [b]_{\alpha} e^{\gamma(|g|_0 + T|a|_0)} + \big(1 + |b|_0^{1+\alpha} e^{\gamma(1+\frac{1}{2}\alpha)(T|a|_0 + |g|_0)} + |h|_{\alpha}^{2+\alpha}\big)|u|_0. \tag{3.10}$$

Proof. We start by setting $F(t,x,y) = a(t,x) - b(t,x)\exp(\gamma y)$, and note that, by Lemma A.6, $F(\cdot,\cdot,w(\cdot,\cdot))$ is in $C^{\alpha}([0,T]\times\mathbb{R})$, whenever w is. Therefore, by Lemma 3.1, for each $w \in C^{\alpha}([0,T]\times\mathbb{R})$, there exists a unique $C^{2+\alpha}([0,T]\times\mathbb{R})$-solution u of the Cauchy problem

$$\begin{cases} 0 = u_t + \frac{1}{2} u_{xx} + h u_x + F(\cdot,\cdot,w(\cdot,\cdot)) \text{ on } [0,T) \times \mathbb{R}, \\ u(T,\cdot) = g(\cdot). \end{cases} \tag{3.11}$$

Consequently, due to Lemma 3.1, the operator $\mathcal{G} : C^{\alpha}([0,T]\times\mathbb{R}) \to C^{2+\alpha}([0,T]\times\mathbb{R})$ which assigns to $w \in C^{\alpha}([0,T]\times\mathbb{R})$ the unique solution of (3.11) is well-defined. For $u_1, u_2 \in C^{\alpha}([0,T]\times\mathbb{R})$, the function $\kappa = \mathcal{G}u_1 - \mathcal{G}u_2$ satisfies

$$\begin{cases} 0 = \kappa_t + \frac{1}{2}\kappa_{xx} + h\kappa_x + \delta \text{ on } [0,T)\times\mathbb{R}, \\ \kappa(T,\cdot) = 0. \end{cases} \tag{3.12}$$

where $\delta(t,x) = F(t,x,u_1(t,x)) - F(t,x,u_2(t,x))$. The inequality (3.2) of Lemma 3.1 implies that

$$|\kappa|_{(\beta)} = |\mathcal{G}u_2 - \mathcal{G}u_1|_{(\beta)} \leq \tfrac{1}{\beta} |\delta|_{(\beta)}. \tag{3.13}$$

On the other hand, using inequality (A.10) of Lemma A.6, we obtain

$$|\delta|_{(\beta)} \leq \beta e^{-\beta(T-t)} \left|b(t,x)\right| \gamma e^{\gamma \lceil u_1 \rceil \vee \lceil u_2 \rceil} \left|u_2(t,x) - u_1(t,x)\right|_0, \tag{3.14}$$

where we remind the reader that $\lceil u \rceil = \sup_{(t,x)\in[0,T]\times\mathbb{R}} u(t,x)$.

If u belongs to the range $\mathcal{RG}$ of $\mathcal{G}$, then the maximum principle applied to equation (3.11) and the non-negativity of b imply that

$$\lceil u \rceil \leq T|a|_0 + |g|_0. \tag{3.15}$$

Inequalities (3.13), (3.14) and (3.15) show that for $u_1, u_2 \in \mathcal{RG}$

$$|\mathcal{G}u_2 - \mathcal{G}u_1|_{(\beta)} \leq \frac{1}{\beta} |b|_0 \gamma e^{\gamma(T|a|_0 + |g|_0)} |u_2 - u_1|_{(\beta)}. \tag{3.16}$$

It follows that the mapping $\mathcal{G}$ is a $|\cdot|_{(\beta)}$-contraction for $\beta > 0$ large enough. For such β, the sequence $\{u_n\}_{n\in\mathbb{N}}$, where $u_1 \in C^\alpha([0,T]\times\mathbb{R})$ and $u_{n+1} = \mathcal{G}u_n$, $n \in \mathbb{N}$, converges towards some $\hat{u} \in C([0,T]\times\mathbb{R})$ in $|\cdot|_{(\beta)}$, and, therefore, also in $|\cdot|_0$. Our next task is to show that $\hat{u}$ is not only in $C([0,T]\times\mathbb{R})$, but also in $C^\alpha([0,T]\times\mathbb{R})$ and that it is, indeed, a fixed point of $\mathcal{G}$.

Let $u \in \mathcal{R}\mathcal{G}$ and set $\tilde{u} = \mathcal{G}u$. Then, the inequality (3.15) is satsified by $\tilde{u}$ as well. Moreover, applying the maximum principle once again to equation (3.11), we get that

$$-|g|_0 - T|b|_0 e^{\gamma(T|a|_0+|g|_0)} \le \tilde{u}(t,x). \tag{3.17}$$

Combining (3.17) with inequality (3.15) for $\tilde{u}$, we obtain

$$|\tilde{u}|_0 \le |g|_0 + T\max\{|a|_0, |b|_0 e^{\gamma(T|a|_0+|g|_0)}\}. \tag{3.18}$$

which provides a uniform $|\cdot|_0$-bound on all elements of the sequence $\{u_n\}_{n\in\mathbb{N}}$. In the remainder of the proof, D will denote a generic constant which may depend on $\alpha, T, \gamma, |h|_\alpha, |a|_\alpha, |b|_\alpha$ or $|g|_{2+\alpha}$, but is independent of n and u_1 and may change from occurrence to occurrence. Thanks to the uniform bound on $|u_n|_0$ established in (3.18) and recalling that $u_{n+1} = \mathcal{G}u_n \in C^{2+\alpha}([0,T]\times\mathbb{R})$, the first inequality in (A.5) of Theorem A.3 yields

$$[u_{n+1}]_\alpha \le D[u_{n+1}]_{2+\alpha}^{\alpha/(2+\alpha)}. \tag{3.19}$$

On the other hand, Corollary A.5 implies that

$$[u_{n+1}]_{2+\alpha} \le D\Big([g]_{2+\alpha} + (1+|h|_\alpha^{2+\alpha})|u_{n+1}|_0 + [a - be^{\gamma u_n}]_\alpha\Big), \tag{3.20}$$

while by inequality (A.9) of Lemma A.6, we have $[e^{\gamma w}]_\alpha \le \gamma e^{\gamma\lceil w\rceil}[w]_\alpha$ for any $w \in C^\alpha([0,T]\times\mathbb{R})$. So, using once more the uniform bound (3.18) on $|u_n|_0$, we get

$$[u_{n+1}]_{2+\alpha} \le D(1 + |b|_\alpha |e^{\gamma u_n}|_\alpha) \le D(1 + [u_n]_\alpha).$$

Therefore, by (3.19), we have

$$[u_{n+1}]_\alpha \le D(1 + [u_n]_\alpha)^{\alpha/(2+\alpha)}, \text{ for } n \in \mathbb{N}.$$

This fact implies that the sequence $x_n = [u_n]_\alpha^{(2+\alpha)/\alpha}$ satisfies (3.5) with $A = B = D^{(2+\alpha)/\alpha}$ so that by Lemma 3.2,

$$\sup_n [u_n]_\alpha < \infty.$$

It is not difficult to see that the $|\cdot|_0$-closure of a $|\cdot|_\alpha$-bounded subset of $C([0,T]\times\mathbb{R})$ is, in fact, a subset of $C^\alpha([0,T]\times\mathbb{R})$. Consequently, the $|\cdot|_0$-limit $\hat{u}$ of $\{u_n\}_{n\in\mathbb{N}}$ is in $C^\alpha([0,T]\times\mathbb{R})$, and in particular, $\mathcal{G}\hat{u}$ is well-defined. Inequality (3.16) yields

$$|\mathcal{G}\hat{u} - u_{n+1}|_0 = |\mathcal{G}\hat{u} - \mathcal{G}u_n|_0 \le D|\hat{u} - u_n|_0,$$

so that $\mathcal{G}\hat{u} = \lim_n u_n = \hat{u}$, i.e., $\hat{u}$ solves (3.8). Moreover, any solution u must satisfy (3.15) and the relation (3.16) guarantees that $\hat{u}$ is unique in the class $C^\alpha([0,T]\times\mathbb{R})$.

To establish the bounds in (3.9) we pick two continuous functions $\hat{a}, \hat{b} : [0,T] \to \mathbb{R}$ and consider the function

$$w(t,x) = A(t) - \frac{1}{\gamma}\log\Big[\gamma B(t) + \exp(-\gamma G)\Big],$$

where $A(t) = \int_t^T \hat{a}(u)\, du$, $B(t) = \int_t^T \hat{b}(u) e^{\gamma A(u)}\, du \geq 0$, and G is an arbitrary constant. As the reader can easily check, we have

$$\begin{cases} 0 = w_t + \frac{1}{2} w_{xx} + h(t,x) w_x + a(t,x) - b(t,x) e^{\gamma w} \\ \qquad - (a(t,x) - \hat{a}(t)) - (\hat{b}(t) - b(t,x)) e^{\gamma w(t,x)} \\ w(T,\cdot) = G. \end{cases}$$

For different choices of functions $\hat{a}, \hat{b}$ and the constant G, w will be either a sub- or a supersolution of (3.8). Indeed, for $G = |g|_0$, $\hat{a}(t) = |a|_0$ and $\hat{b}(t) = 0$, w becomes a supersolution, yielding the upper bound in (3.9). Similarly, for $G = -|g|_0$, $\hat{a}(t) = 0$ and $\hat{b}(t) = |b|_0$, w is a subsolution, and so, the lower bound in (3.9) holds, too.

The last item on our list is the $[\cdot]_{2+\alpha}$-bound (3.10). Estimate (A.7), along with Lemma A.6 and the just established (3.9), yields

$$\begin{aligned} [u]_{2+\alpha} &\preceq [g]_{2+\alpha} + [a]_\alpha + [b e^{\gamma u}]_\alpha + (1 + |h|_\alpha^{2+\alpha}) |u|_0 \\ &\preceq [g]_{2+\alpha} + [a]_\alpha + [b]_\alpha e^{\gamma \lceil u \rceil} + |b|_0 [e^{\gamma u}]_\alpha + (1 + |h|_\alpha^{2+\alpha}) |u|_0 \\ &\preceq [g]_{2+\alpha} + [a]_\alpha + [b]_\alpha e^{\gamma (T|a|_0 + |g|_0)} + |b|_0 \gamma e^{\gamma (T|a|_0 + |g|_0)} [u]_\alpha + (1 + |h|_\alpha^{2+\alpha}) |u|_0. \end{aligned}$$

The inequality (3.10) now follows directly from Corollary A.2. □

Finally, we integrate Lemma 3.1 and Proposition 3.3 and discuss the difficulties in the standard verification argument.

Proposition 3.4. *For $\lambda \in C^\alpha([0,T] \times \mathbb{R} \times \{0,1\})$ and $g \in C^{2+\alpha}(\mathbb{R} \times \{0,1\})$ there exists a unique function $u^{(\lambda)} \in C^{2+\alpha}([0,T] \times \mathbb{R} \times \{0,1\})$ which solves the Cauchy problem (2.4). Moreover, the stochastic process $\{\pi_t^{(\lambda)}\}_{t \in [0,T]}$ with Markov representative $\pi^{(\lambda)} = \frac{1}{\gamma} \lambda - u_x^{(\lambda)}$ is in $\mathcal{A}$ and attains the maximum in the utility maximization problem* (2.2).

Proof. We note, first, that the $n = 1$-slice of the equation (2.4) has the form (3.1) with $h(t,x) = -\lambda(t,x) \in C^\alpha([0,T] \times \mathbb{R})$, $a(t,x) = \frac{1}{2\gamma} \lambda^2(t,x) \in C^\alpha([0,T] \times \mathbb{R})$. By Lemma 3.1, it admits a unique $C^{2+\alpha}([0,T] \times \mathbb{R})$-solution $u(\cdot,\cdot,1)$. Therefore, the $n = 0$-slice is of the form (3.8) with $h(t,x) = -\lambda(t,x)$, $a(t,x) = \frac{1}{\gamma}(\frac{1}{2}\lambda^2(t,x) + \mu)$ and $b(t,x) = \frac{\mu}{\gamma} e^{-\gamma u(t,x,1)}$; the assumptions placed on λ and the $C^\alpha([0,T] \times \mathbb{R})$-property of $u(\cdot,\cdot,1)$ imply that Proposition 3.3 can be applied. Hence, (2.4) admits a unique solution $u^{(\lambda)} \in C^{2+\alpha}([0,T] \times \mathbb{R} \times \{0,1\})$.

Having established existence and regularity of the solution u of (2.4), we turn to the second statement - namely, that the process $\pi^{(\lambda)}$, defined in the statement, is optimal for (2.2). First, we consider the function $v : [0,T] \times \mathbb{R} \times \mathbb{R} \times \{0,1\} \to (-\infty, 0)$, given by

$$v(t,\xi,x,n) = -e^{-\gamma \left(\xi + u^{(\lambda)}(t,x,n) \right)}.$$

The reader will easily check that v is a classical solution of the following PDDE - the formal HJB equation for the utility-maximization problem (2.2):

$$\begin{cases} 0 = v_t + \sup_{\pi \in \mathbb{R}} \left(\frac{1}{2} v_{xx} + \pi \lambda v_\xi + \frac{1}{2} \pi^2 v_{\xi\xi} + \pi v_{x\xi} + \mu v_n \right) \\ v(T,\cdot,\cdot) = -e^{-\gamma \left(\xi + g(x,n) \right)}, \end{cases} \tag{3.21}$$

with all regularity inherited from $u^{(\lambda)}$. The equation (3.21) is the Hamilton-Jacobi-Bellman equation for the control problem (2.2), where the variables ξ, x, n correspond to the wealth process $\int_0^\cdot \pi_u^{(\lambda)}\, dS_u^{(\lambda)}$, the Brownian motion B and the jump process N, respectively. The standard verification procedure (see, e.g., the ideas in the proof of Theorem 8.1, p. 141 in [12]) can be used to show that v is indeed the value function of the utility-maximization problem and that the form of the optimal portfolio can be recognized as the optimal value of the parameter π in the maximization in (3.21). The so-obtained π is admissible since it is uniformly bounded. The usual difficulty one encounters in the course of the verification procedure - namely, the one involved in establishing the martingale property of certain local martingales - is easily dealt with here. One simply needs to observe that v and $u_x^{(\lambda)}$ are uniformly bounded and use the square-integrability of $\pi \in \mathcal{A}$. □

3.2. **Stability of the optimal portfolio.** Having established the existence and uniqueness of the regular solution to (2.4), we turn to the study of the map $\lambda \mapsto u^{(\lambda)}$ in order to prove the statements in the second part of Theorem 2.8. We start with some preliminary growth estimates.

Proposition 3.5. *Let $u \in C^{2+\alpha}([0,T] \times \mathbb{R} \times \{0,1\})$ be the unique solution to (2.4). Define*

$$M_0 = \gamma |g|_0 + \tfrac{1}{2} T |\lambda|_0^2 + \mu T, \quad M_\alpha = \gamma |g|_{2+\alpha} + (T+1)(\tfrac{1}{2}|\lambda|_\alpha^2 + \mu). \tag{3.22}$$

Then,

$$|u^{(\lambda)}|_0 \preceq M_0, \ \textit{and}\ [u^{(\lambda)}]_{2+\alpha} \preceq M_\alpha^{2+\alpha/2} e^{(2+\alpha)M_0}. \tag{3.23}$$

Proof. Set $L_0 = |\lambda|_0$, $L_\alpha = |\lambda|_\alpha$ and $G = |g|_{2+\alpha}$, and let M_0, M_α be as in (3.22). We start with the $n = 1$-slice first; to make the notation more palatable, we write simply $\bar{u}$ for $u^{(\lambda)}(\cdot,\cdot,1)$ and $\bar{\lambda}$ for $\lambda(\cdot,\cdot,1)$ so that $\bar{u}$ satisfies the linear equation (3.1) with $a = \frac{1}{2\gamma}\bar{\lambda}^2$ and $h = -\bar{\lambda}$. With $\overline{g} = g(\cdot, 1)$, the maximum principle implies that

$$|\bar{u}|_0 \le T|a|_0 + |\bar{g}|_0 \le \tfrac{1}{2\gamma} T L_0^2 + |g|_0 \preceq M_0. \tag{3.24}$$

Corollary A.5 yields

$$\begin{aligned} [\bar{u}]_{2+\alpha} &\preceq G + \tfrac{1}{2\gamma}[\bar{\lambda}^2]_\alpha + \big(1 + |\bar{\lambda}|_\alpha^{2+\alpha}\big)|\bar{u}|_0 \\ &\preceq G + L_\alpha^2 + (1 + L_\alpha^{2+\alpha})(\tfrac{1}{2} T L_\alpha^2 + \gamma G) \\ &\preceq (1 + L_\alpha^{2+\alpha})(G + (1+T)L_\alpha^2) \preceq M_\alpha (1 + L_\alpha^{2+\alpha}) \preceq M_\alpha^{2+\alpha/2}, \end{aligned} \tag{3.25}$$

where the last inequality is a consequence of the fact that the function $\rho \mapsto (1 + x^\rho)^{1/\rho}$ is nonincreasing in ρ for $x \ge 0$ and $\rho > 0$. Also, due to the first inequality in (A.5) and the just established (3.24) and (3.25), we have

$$[\bar{u}]_\alpha \preceq [\bar{u}]_{2+\alpha}^{\frac{\alpha}{2+\alpha}} |\bar{u}|_0^{\frac{2}{2+\alpha}} \preceq (M_\alpha^{2+\frac{\alpha}{2}})^{\frac{\alpha}{2+\alpha}} M_0^{\frac{2}{2+\alpha}} \preceq M_\alpha^{2+\frac{\alpha}{2}}. \tag{3.26}$$

We move on to the $n = 0$-slice, and write $\underline{u}(\cdot,\cdot)$ for $u^{(\lambda)}(\cdot,\cdot,0)$ and $\underline{\lambda}$ for $\lambda(\cdot,\cdot,0)$ so that $\underline{u}$ satisfies the semi-linear Cauchy problem (3.8) with $a = \frac{1}{2\gamma}\underline{\lambda}^2 + \frac{\mu}{\gamma}$, $h = -\underline{\lambda}$ and $b = \frac{\mu}{\gamma}\exp(-\gamma \bar{u})$. Note that $\bar{u}$ also solves equation (3.8) with $a = \frac{1}{2\gamma}\bar{\lambda}^2, h = -\bar{\lambda}$ and $b = 0$, so that the estimate (3.9)

implies $|e^{-\gamma\bar{u}}|_0 = e^{-\gamma\lfloor\bar{u}\rfloor} \le e^{\gamma|\bar{g}|_0}$, as well as

$$\begin{aligned}|\underline{u}|_0 &\le \max\Big(\tfrac{1}{2\gamma}T(L_0^2+2\mu)+|g|_0, \tfrac{1}{\gamma}\log(e^{\gamma|g|_0}+\mu T|e^{-\gamma\bar{u}}|_0)\Big)\\ &\le |g|_0 + \tfrac{1}{\gamma}\max\Big(T(\tfrac{1}{2}L_0^2+\mu), \log(1+\mu T)\Big).\end{aligned}$$

This result, together with (3.24) and the fact that $\mu T \ge \log(1+\mu T)$, yields the $|\cdot|_0$-bound in (3.23). By inequality (A.9) of Lemma A.6 and estimates (3.9) and (3.26), we have

$$[e^{-\gamma\bar{u}}]_\alpha \le \gamma e^{\gamma G} M_\alpha^{2+\frac{\alpha}{2}}. \tag{3.27}$$

Hence, using again the estimate (3.9) for $\bar{u}$, the just established first inequality in (3.23) and (3.27), we obtain

$$[e^{-\gamma\bar{u}}e^{\gamma\underline{u}}]_\alpha \preceq e^{\gamma G}e^{M_0}M_\alpha^{2+\frac{\alpha}{2}} + e^{\gamma G}e^{M_0}[\underline{u}]_\alpha = e^{\gamma G+M_0}(M_\alpha^{2+\frac{\alpha}{2}}+[\underline{u}]_\alpha).$$

So, due to Corollary A.5,

$$\begin{aligned}[\underline{u}]_{2+\alpha} &\preceq [\underline{g}]_{2+\alpha} + \frac{1}{2\gamma}[\underline{\lambda}^2]_\alpha + \tfrac{\mu}{\gamma}[e^{-\gamma\bar{u}}e^{\gamma\underline{u}}]_\alpha + (1+|\underline{\lambda}|_\alpha^{2+\alpha})|\underline{u}|_0\\ &\preceq \tfrac{1}{\gamma}(\gamma G+\tfrac{1}{2}L_\alpha^2+\mu) + [e^{-\gamma\bar{u}}e^{\gamma\underline{u}}]_\alpha + (1+L_\alpha^{2+\alpha})|\underline{u}|_0\\ &\preceq M_0 + e^{M_0+\gamma G}M_\alpha^{2+\frac{\alpha}{2}} + e^{M_0+\gamma G}[\underline{u}]_\alpha + (1+L_\alpha^{2+\alpha})|\underline{u}|_0.\end{aligned}$$

Therefore, thanks to Corollary A.2 and the first inequality in (3.23),

$$[\bar{u}]_{2+\alpha} \preceq e^{\gamma G+M_0} + \Big(e^{(\gamma G+M_0)(1+\frac{\alpha}{2})} + 1 + L_\alpha^{2+\alpha}\Big)|\underline{u}|_0.$$

Thus, using once more the decrease of the function $\rho \mapsto (1+x^\rho)^{1/\rho}$, $x,\rho>0$ and the first inequality of (3.23), we get

$$[\underline{u}]_{2+\alpha} \preceq e^{(2+\alpha)M_0}M_\alpha^{2+\frac{\alpha}{2}} + M_0 M_\alpha^{2+\frac{\alpha}{2}} \preceq e^{(2+\alpha)M_0}M_\alpha^{2+\frac{\alpha}{2}}.$$

The last inequality, along with (3.25), completes the proof. □

The multiplicative interpolation inequalities of Theorem A.3 yield the following corollary:

Corollary 3.6. *Let $u^{(\lambda)} \in C^{2+\alpha}([0,T]\times\mathbb{R}\times\{0,1\})$ be the unique solution to* (2.4)*, and let M_0 and M_α be as in* (3.22)*. Then, the following estimates hold*

$$\begin{aligned}&[u^{(\lambda)}]_\alpha \preceq M_\alpha^{2+\alpha/2}e^{\alpha M_0}, && |u_x^{(\lambda)}|_0 \preceq M_\alpha^{2+\alpha/2}e^{M_0}\\ &[u_x^{(\lambda)}]_\alpha \preceq M_\alpha^{2+\alpha/2}e^{(1+\alpha)M_0}, && |u_x^{(\lambda)}|_\alpha \preceq M_\alpha^{2+\alpha/2}e^{(1+\alpha)M_0}.\end{aligned} \tag{3.28}$$

We continue with a stability estimate for the nonlinear, $n=1$-slice.

Proposition 3.7. *For $g \in C^{2+\alpha}(\mathbb{R})$, $a^{(k)} \in C^\alpha([0,T]\times\mathbb{R})$, $h^{(k)} \in C^\alpha([0,T]\times\mathbb{R})$ and $b^{(k)} \in C^\alpha([0,T]\times\mathbb{R})$, let $u^{(k)}$, $k=1,2$ be the unique $C^{2+\alpha}([0,T]\times\mathbb{R})$-solution to*

$$\begin{cases}0 = u_t^{(k)} + \frac{1}{2}u_{xx}^{(k)} + h^{(k)}u_x^{(k)} + a^{(k)} - b^{(k)}e^{\gamma u^{(k)}} \text{ on } [0,T)\times\mathbb{R},\\ u^{(k)}(T,\cdot) = g,\end{cases} \tag{3.29}$$

and let

$$D = 1+\max\Big(|b^{(1)}|_\alpha^{1+\alpha/2}, |u_x^{(1)}|_\alpha, |u^{(2)}|_\alpha, |h^{(2)}|_\alpha^{2+\alpha}\Big),\ P = \gamma e^{\gamma\lceil u^{(1)}\rceil \vee \lceil u^{(2)}\rceil}.$$

Then,

$$|u^{(2)} - u^{(1)}|_0 \leq T e^{T|b^{(1)}|_0 P}\Big(|a^{(\delta)}|_0 + D|h^{(\delta)}|_0 + P|b^{(\delta)}|_0\Big), \text{ and} \tag{3.30}$$

$$[u^{(2)} - u^{(1)}]_{2+\alpha} \preceq [a^{(\delta)}]_\alpha + D|h^{(\delta)}|_\alpha + PD|b^{(\delta)}|_\alpha + P^{1+\alpha/2}D|u^{(2)} - u^{(1)}|_0, \tag{3.31}$$

where $a^{(\delta)} = a^{(2)} - a^{(1)}, h^{(\delta)} = h^{(2)} - h^{(1)}$ *and* $b^{(\delta)} = b^{(2)} - b^{(1)}$.

Proof. Set $u^{(\delta)} = u^{(2)} - u^{(1)}$ and subtract the equations (3.29) with $k = 2$ and $k = 1$, respectively, to obtain that $u^{(\delta)}$ is a $C^{2+\alpha}([0,T] \times \mathbb{R})$-solution to the following semilinear Cauchy problem

$$\begin{cases} 0 = u^{(\delta)}_t + \frac{1}{2}u^{(\delta)}_{xx} + h^{(2)}u^{(\delta)}_x + h^{(\delta)}u^{(1)}_x + a^{(\delta)} \\ \qquad\qquad - b^{(\delta)}e^{\gamma u^{(2)}} - b^{(1)}(e^{\gamma u^{(2)}} - e^{\gamma u^{(1)}}), \text{ on } [0,T) \times \mathbb{R}, \\ u^{(\delta)}(T,\cdot) = 0. \end{cases} \tag{3.32}$$

To alleviate the notation, we set

$$K_0 = |h^{(\delta)}u^{(1)}_x + a^{(\delta)} - b^{(\delta)}e^{\gamma u^{(2)}}|_0 \text{ and } K_\alpha = [h^{(\delta)}u^{(1)}_x + a^{(\delta)} - b^{(\delta)}e^{\gamma u^{(2)}}]_\alpha.$$

First, we establish the $|\cdot|_0$-bound in (3.30). Let $U(t) = \sup_{x\in\mathbb{R}} \big|u^{(\delta)}(t,x)\big|$ so that, by the mean value theorem, we have

$$\Big|e^{\gamma u^{(2)}(t,x)} - e^{\gamma u^{(1)}(t,x)}\Big| \leq P\,U(t), \text{ for all } (t,x) \in [0,T] \times \mathbb{R}.$$

Hence, the maximum principle with equation (3.32) implies

$$\Big|u^{(\delta)}(t,x)\Big| \leq \int_t^T \big(K_0 + |b^{(1)}|_0 P\,U(s)\big)\,ds. \tag{3.33}$$

We can, therefore, employ Gronwall's inequality (see, e.g., [27], p. 543) to obtain

$$U(t) \leq (T-t)K_0 e^{(T-t)|b^{(1)}|_0 P} \leq T K_0 e^{T|b^{(1)}|_0 P}.$$

Noting that $K_0 \leq D|h^{(\delta)}|_0 + |a^{(\delta)}|_0 + P|b^{(\delta)}|_0$, we conclude that (3.30) holds.

Turning to the $[\cdot]_{2+\alpha}$-bound (3.31), we use Corollary A.5 and Lemma A.6 to conclude that

$$\begin{aligned} [u^{(\delta)}]_{2+\alpha} &\preceq [h^{(\delta)}u^{(1)}_x + a^{(\delta)} - b^{(\delta)}e^{\gamma u^{(2)}} - b^{(1)}(e^{\gamma u^{(2)}} - e^{\gamma u^{(1)}})]_\alpha + \Big(1 + |h^{(2)}|^{2+\alpha}_\alpha\Big)|u^{(\delta)}|_0 \\ &\preceq K_\alpha + |b^{(1)}|_0[e^{\gamma u^{(2)}} - e^{\gamma u^{(1)}}]_\alpha + [b^{(1)}]_\alpha|e^{\gamma u^{(2)}} - e^{\gamma u^{(1)}}|_0 + D|u^{(\delta)}|_0 \\ &\preceq K_\alpha + |b^{(1)}|_0 P[u^{(\delta)}]_\alpha + P|u^{(2)}|_\alpha|u^{(\delta)}|_0 + [b^{(1)}]_\alpha P|u^{(\delta)}|_0 + D|u^{(\delta)}|_0. \end{aligned} \tag{3.34}$$

Therefore, by Corollary A.2, we have

$$\begin{aligned} [u^{(\delta)}]_{2+\alpha} &\preceq K_\alpha + \Big(P|b^{(1)}|_0)^{1+\alpha/2} + P|u^{(2)}|_\alpha + P[b^{(1)}]_\alpha + D\Big)|u^{(\delta)}|_0 \\ &\preceq K_\alpha + P^{1+\alpha/2}D|u^{(\delta)}|_0. \end{aligned} \tag{3.35}$$

Finally, due to Lemma A.6, $K_\alpha \preceq D|h^{(\delta)}|_\alpha + [a^{(\delta)}]_\alpha + PD|b^{(\delta)}|_\alpha$, which implies (3.31). □

In conjunction with the multiplicative interpolation inequalities of Theorem A.3, our final result, Proposition 3.8, yields the remaining statements of Theorem 2.8.

Proposition 3.8. *For $g \in C^{2+\alpha}([0,T]\times\mathbb{R}\times\{0,1\})$ and $k=1,2$, let $u^{(k)} \in C^{2+\alpha}([0,T]\times\mathbb{R}\times\{0,1\})$ be the unique solution to (2.4), corresponding to $\lambda = \lambda^{(k)} \in C^{\alpha}([0,T]\times\mathbb{R}\times\{0,1\})$. Then,*

$$|u^{(2)} - u^{(1)}|_0 \preceq T\, M_\alpha^{2+2\alpha} e^{e^{2M_0+1}} |\lambda^{(2)} - \lambda^{(1)}|_0, \text{ and}$$
$$[u^{(2)} - u^{(1)}]_{2+\alpha} \preceq M_\alpha^{7+3\alpha} e^{e^{2M_0+2}} |\lambda^{(2)} - \lambda^{(1)}|_\alpha,$$

where

$$M_0 = \gamma |g|_0 + \mu T + \tfrac{1}{2} T \max(|\lambda^{(1)}|_0, |\lambda^{(2)}|_0)^2, \text{ and}$$
$$M_\alpha = |g|_{2+\alpha} + (1+T)\Big(\mu + \tfrac{1}{2}\max(|\lambda^{(1)}|_\alpha, |\lambda^{(2)}|_\alpha)^2\Big).$$

Proof. Let $L_0 = \max(|\lambda^{(1)}|_0, |\lambda^{(2)}|_0)$, $L_\alpha = \max((|\lambda^{(1)}|_\alpha, |\lambda^{(2)}|_\alpha)$, $G_0 = |g|_0$ and $G = |g|_{2+\alpha}$. Then,

$$M_0 = \gamma G_0 + \mu T + \tfrac{1}{2} T L_0^2,\ M_\alpha = G + (1+T)(1+L_\alpha^2).$$

Just like in the proof of Proposition 3.5, we deal with the $n=1$-slice first; again, to make the notation more palatable, we write $\bar{u}^{(k)}$ for $u^{(k)}(\cdot,\cdot,1)$ and $\bar{\lambda}^{(k)}$ for $\lambda^{(k)}(\cdot,\cdot,1)$ so that $\bar{u}^{(k)}$ satisfies the linear equation (3.29) with $a^{(k)} = \frac{1}{2\gamma}(\bar{\lambda}^{(k)})^2$, $h^{(k)} = -\bar{\lambda}^{(k)}$ and $b^{(k)} = 0$. Also, set $\bar{u}^{(\delta)} = \bar{u}^{(2)} - \bar{u}^{(1)}$, $\Delta_0 = |\lambda^{(2)} - \lambda^{(1)}|_0$ and $\Delta_\alpha = |\lambda^{(2)} - \lambda^{(1)}|_\alpha$. Then, $\bar{u}^{(\delta)}$ solves the Cauchy problem (3.32) with $h^{(\delta)} = -\bar{\lambda}^{(2)} + \bar{\lambda}^{(1)}$, $a^{(\delta)} = \frac{1}{2\gamma}((\bar{\lambda}^{(2)})^2 - (\bar{\lambda}^{(1)})^2)$ and $b^{(\delta)} = 0$. Inequality (3.33) and Corollary 3.6 yield

$$|\bar{u}^{(\delta)}|_0 \preceq T(L_0\Delta_0 + |\bar{u}_x^{(1)}|_0 \Delta_0) \preceq T\Delta_0(L_0 + M_\alpha^{2+\alpha/2} e^{M_0}) \preceq T M_\alpha^{2+\alpha/2} e^{M_0} \Delta_0. \tag{3.36}$$

So, by Corolary A.5, estimate (3.28) and the decrease of the function $\rho \mapsto (1+x^\rho)^{1/\rho}$, we have

$$\begin{aligned}[\bar{u}^{(\delta)}]_{2+\alpha} &\preceq |\bar{u}_x^{(1)}|_\alpha \Delta_\alpha + \Delta_\alpha L_\alpha + (1 + L_\alpha^{2+\alpha}) |\bar{u}^{(\delta)}|_0 \\ &\preceq \Big(M_\alpha^{2+\alpha/2} e^{(1+\alpha)M_0} + M_\alpha + M_\alpha^{1+\alpha/2} T M_\alpha^{2+\alpha/2} e^{M_0}\Big)\Delta_\alpha \\ &\preceq M_\alpha^{4+\alpha} e^{(1+\alpha)M_0} \Delta_\alpha.\end{aligned} \tag{3.37}$$

Consequently, the multiplicative interpolation inequalities of Theorem A.3 imply

$$[\bar{u}^{(\delta)}]_\alpha \preceq T^{2/(2+\alpha)} M_\alpha^{2+2\alpha} e^{(1+\alpha)M_0} \Delta_\alpha. \tag{3.38}$$

We proceed with the $n=0$-slice, and write $\underline{u}^{(k)}(\cdot,\cdot)$ for $u^{(k)}(\cdot,\cdot,0)$ and $\underline{\lambda}^{(k)}$ for $\lambda^{(k)}(\cdot,\cdot,0)$ so that $\underline{u}^{(k)}$ satisfies the semi-linear Cauchy problem (3.8) with $a^{(k)} = \frac{1}{2\gamma}(\underline{\lambda}^{(k)})^2 + \frac{\mu}{\gamma}$, $h^{(k)} = -\underline{\lambda}^{(k)}$ and $b^{(k)} = \frac{\mu}{\gamma}\exp(-\gamma \bar{u}^{(k)})$. Set

$$D = M_\alpha^{2+2\alpha} e^{(1+2\alpha)M_0} \text{ and } P = \gamma e^{\gamma \lceil g \rceil + \frac{1}{2} T L_0^2},$$

so that, by Proposition 3.5, Corollary 3.6 and inequality (3.15), D and P dominate (in the $\preceq$ $-$sense) the D and P of Proposition 3.7 with the current choice of $b^{(1)}, u^{(1)}, u^{(2)}$ and $h^{(2)}$.

Set $\underline{u}^{(\delta)} = \underline{u}^{(2)} - \underline{u}^{(1)}$. Then $\underline{u}^{(\delta)}$ satisfies the equation (3.32) with $u^{(1)} = \underline{u}^{(1)}$, $h^{(2)} = -\underline{\lambda}^{(2)}$, $h^{(\delta)} = -\underline{\lambda}^{(2)} + \underline{\lambda}^{(1)}$, $a^{(\delta)} = \frac{1}{2\gamma}((\underline{\lambda}^{(2)})^2 - (\underline{\lambda}^{(1)})^2)$, $b^{(\delta)} = \frac{\mu}{\gamma}(\exp(-\gamma\bar{u}^{(2)}) - \exp(-\gamma\bar{u}^{(1)}))$ and $b^{(1)} = \frac{\mu}{\gamma}\exp(-\gamma\bar{u}^{(1)})$. Inequality (3.30) of Proposition 3.7 and the lower bound on $\bar{u}^{(1)}$ in (3.9) give us

$$\begin{aligned}|\underline{u}^{(\delta)}|_0 &\preceq T e^{T|b^{(1)}|_0 P}\big(\tfrac{1}{2\gamma}|(\underline{\lambda}^{(2)})^2 - (\underline{\lambda}^{(1)})^2|_0 + D|-\underline{\lambda}^{(2)} + \underline{\lambda}^{(1)}|_0 + P|e^{-\gamma\bar{u}^{(1)}} - e^{-\gamma\bar{u}^{(2)}}|_0\big) \\ &\preceq T e^{T\mu e^{2\gamma G_0 + \frac{1}{2}TL_0^2}}(\Delta_0 L_0 + D\Delta_0 + P|e^{-\gamma\bar{u}^{(1)}} - e^{-\gamma\bar{u}^{(2)}}|_0).\end{aligned}$$

On the other hand, thanks to Lemma A.6, the lower bound on $\bar{u}^{(1)}$ and $\bar{u}^{(2)}$ in (3.9) and inequality (3.36), we have

$$|e^{-\gamma\bar{u}^{(2)}} - e^{-\gamma\bar{u}^{(1)}}|_0 \preceq \gamma e^{\gamma G_0}|\bar{u}^{(\delta)}|_0 \preceq \gamma e^{\gamma G_0} T M_\alpha^{2+\alpha/2} e^{M_0}\Delta_0.$$

Combining the last two inequalities we, obtain

$$\begin{aligned}
|\underline{u}^{(\delta)}|_0 &\preceq T e^{T\mu e^{2\gamma G_0+\frac{1}{2}TL_0^2}}(L_0 + D + Pe^{\gamma G_0}TM_\alpha^{2+\alpha/2}e^{M_0})\Delta_0\\
&\preceq T e^{T\mu e^{2\gamma G_0+\frac{1}{2}TL_0^2}}(M_\alpha^{2+2\alpha}e^{(1+2\alpha)M_0} + Pe^{\gamma G_0}TM_\alpha^{2+\alpha/2}e^{M_0})\Delta_0\\
&\preceq T e^{T\mu e^{2\gamma G_0+\frac{1}{2}TL_0^2}}M_\alpha^{2+2\alpha}e^{(1+2\alpha)M_0}(1 + e^{2\gamma G_0+\frac{1}{2}TL_0^2}T)\Delta_0\\
&\preceq T(1+T) e^{T\mu e^{2\gamma G_0+\frac{1}{2}TL_0^2}}M_\alpha^{2+2\alpha}e^{(1+2\alpha)M_0}e^{2\gamma G_0+\frac{1}{2}TL_0^2}\Delta_0\\
&\preceq TM_\alpha^{2+2\alpha}e^A\Delta_0,
\end{aligned} \tag{3.39}$$

where $A = \log(1+\mu T) + M_0 + \mu T e^{2\gamma G_0 + 1/2TL_0^2} + (1+2\alpha)M_0 + 2\gamma G_0 + \frac{1}{2}TL_0^2$. Since $\frac{x}{e^x} \leq \frac{1}{e}$, for all $x \geq 0$ and $\log(1+\mu T) \leq \mu T$, we have

$$\begin{aligned}
A &= \log(1+\mu T) + M_0 + \mu T e^{\gamma G_0 + M_0 - \mu T} + (1+2\alpha)M_0 + M_0 + \gamma G_0 - \mu T\\
&\preceq M_0 + e^{\gamma G_0 + M_0}\tfrac{\mu T}{e^{\mu T}} + (2+2\alpha)M_0 + \gamma G_0 \preceq e^{2M_0+1}.
\end{aligned} \tag{3.40}$$

It remains to estimate the $[\cdot]_{2+\alpha}$-seminorm of $\underline{u}^{(\delta)}$. By (3.24),(3.25) and Theorem A.3, we have

$$1 + [\bar{u}^{(2)}]_\alpha \preceq 1 + (M_\alpha^{2+\alpha/2})^{\alpha/(2+\alpha)}M_0^{2/(2+\alpha)} \preceq 1 + M_\alpha^{1+\alpha} \preceq M_\alpha^{1+\alpha}. \tag{3.41}$$

Applying Corollaries A.5 and 3.6, and using the decrease of the function $\rho \mapsto (1+x^\rho)^{1/\rho}, x, \rho > 0$, we get

$$\begin{aligned}
[\underline{u}^{(\delta)}]_{2+\alpha} &\preceq \Delta_\alpha|\underline{u}_x^{(1)}|_\alpha + L_\alpha\Delta_\alpha + e^{\gamma G_0}(1 + [\underline{u}^{(2)}]_\alpha)|e^{-\gamma\bar{u}^{(2)}} - e^{-\gamma\bar{u}^{(1)}}|_\alpha\\
&\quad + |e^{-\gamma\bar{u}^{(1)}}|_\alpha|e^{\gamma\underline{u}^{(2)}} - e^{\gamma\underline{u}^{(1)}}|_\alpha + (1 + L_\alpha^{2+\alpha})|\underline{u}^{(\delta)}|_0\\
&\preceq \Delta_\alpha M_\alpha^{2+\alpha/2}e^{(1+\alpha)M_0} + M_\alpha\Delta_\alpha + e^{\gamma G_0}(1 + M_\alpha^{2+\alpha/2}e^{\alpha M_0})|e^{-\gamma\bar{u}^{(2)}} - e^{-\gamma\bar{u}^{(1)}}|_\alpha\\
&\quad + |e^{-\gamma\bar{u}^{(1)}}|_\alpha|e^{\gamma\underline{u}^{(2)}} - e^{\gamma\underline{u}^{(1)}}|_\alpha + M_\alpha^{1+\alpha/2}|\underline{u}^{(\delta)}|_0.
\end{aligned} \tag{3.42}$$

On the other hand, by Lemma A.6, the lower bound on $\bar{u}^{(1)}$ and $\bar{u}^{(2)}$ in (3.9), and inequalities (3.36), (3.37) and (3.41), we have

$$\begin{aligned}
|e^{-\gamma\bar{u}^{(2)}} - e^{-\gamma\bar{u}^{(1)}}|_\alpha &\preceq e^{\gamma G_0}\big(|\bar{u}^{(\delta)}|_0(1 + [\bar{u}^{(2)}]_\alpha) + [\bar{u}^{(\delta)}]_\alpha\big)\\
&\preceq e^{\gamma G_0}\big(TM_\alpha^{2+3\alpha/2}e^{M_0} + T^{2/(2+\alpha)}M_\alpha^{2+2\alpha}e^{(1+\alpha)M_0}\big)\Delta_\alpha\\
&\preceq M_\alpha^{4+3\alpha/2}e^{(2+\alpha)M_0}\Delta_\alpha.
\end{aligned} \tag{3.43}$$

Moreover, due to Lemma A.6 and Corollary 3.6, $|e^{-\gamma\bar{u}^{(1)}}|_\alpha \preceq M_\alpha^{2+\alpha/2}e^{(1+\alpha)M_0}$ and, with the upper bound in (3.9) and inequalities (3.39) and (3.40) in mind,

$$\begin{aligned}
|e^{-\gamma\underline{u}^{(2)}} - e^{-\gamma\underline{u}^{(1)}}|_\alpha &\preceq e^{M_0}\big(TM_\alpha^{2+2\alpha}e^{e^{2M_0+1}}\Delta_0(1 + M_\alpha^{2+\alpha/2}e^{\alpha M_0}) + [\bar{u}^{(\delta)}]_\alpha\big)\\
&\preceq M_\alpha^{5+5\alpha/2}e^{e^{2M_0+1}}e^{(1+\alpha)M_0}\Delta_0 + e^{M_0}[\bar{u}^{(\delta)}]_\alpha.
\end{aligned}$$

So,

$$|e^{-\gamma\bar{u}^{(1)}}|_\alpha|e^{-\gamma\underline{u}^{(2)}} - e^{-\gamma\underline{u}^{(1)}}|_\alpha \preceq M_\alpha^{7+3\alpha}e^{e^{2M_0+1}}e^{(2+2\alpha)M_0}\Delta_0 + M_\alpha^{2+\alpha/2}e^{(2+\alpha)M_0}[\bar{u}^{(\delta)}]_\alpha. \tag{3.44}$$

Combining inequalities (3.42), (3.43) and (3.44), we get

$$\begin{aligned}
[\underline{u}^{(\delta)}]_{2+\alpha} \preceq M_\alpha^{2+\alpha/2}e^{(1+\alpha)M_0}\Delta_\alpha + M_\alpha^{2+\alpha/2}e^{(1+\alpha)M_0}M_\alpha^{4+3\alpha/2}e^{(2+\alpha)M_0}\Delta_\alpha \\
+ M_\alpha^{7+3\alpha}e^{e^{2M_0+1}}e^{(2+2\alpha)M_0}\Delta_\alpha + M_\alpha^{2+\alpha/2}e^{(2+\alpha)M_0}[\bar{u}^{(\delta)}]_\alpha + M_\alpha^{1+\alpha/2}|\underline{u}^{(\delta)}|_0 \\
\preceq M_\alpha^{7+3\alpha}e^{e^{2M_0+1}}e^{(3+2\alpha)M_0}\Delta_\alpha + M_\alpha^{2+\alpha/2}e^{(2+\alpha)M_0}[\bar{u}^{(\delta)}]_\alpha + M_\alpha^{1+\alpha/2}|\underline{u}^{(\delta)}|_0.
\end{aligned}$$

Thanks to Corollary A.2, we obtain

$$\begin{aligned}
[\underline{u}^{(\delta)}]_{2+\alpha} &\preceq M_\alpha^{7+3\alpha}e^{e^{2M_0+1}}e^{(3+2\alpha)M_0}\Delta_\alpha + \Big((M_\alpha^{2+\alpha/2}e^{(2+\alpha)M_0})^{1+\alpha/2} + M_\alpha^{1+\alpha/2}\Big)|\underline{u}^{(\delta)}|_0 \\
&\preceq M_\alpha^{7+3\alpha}e^{e^{2M_0+1}}e^{(3+2\alpha)M_0}\Delta_\alpha + M_\alpha^{2+2\alpha}e^{(2+5\alpha/2)M_0}|\underline{u}^{(\delta)}|_0.
\end{aligned}$$

Finally, due to inequality (3.39), we get

$$\begin{aligned}
[\underline{u}^{(\delta)}]_{2+\alpha} &\preceq M_\alpha^{7+3\alpha}e^{e^{2M_0+1}}e^{(3+2\alpha)M_0}\Delta_\alpha + M_\alpha^{5+4\alpha}e^{(2+5\alpha/2)M_0}e^{e^{2M_0+1}}\Delta_\alpha \\
&\preceq M_\alpha^{7+3\alpha}e^{e^{2(M_0+1)}}\Delta_\alpha.
\end{aligned}$$

□

Appendix A. Anisotropic Hölder Spaces

A.1. **Definitions and notation.** Classical (anisotropic) Hölder spaces provide a convenient setting for stability analysis of a class of utility-maximization problems. Here is a short overview of the notation and some basic definitions.

Let $C([0,T]\times\mathbb{R})$ be the set of all continuous functions $u:[0,T]\times\mathbb{R}\to\mathbb{R}$, and let $C_b([0,T]\times\mathbb{R})$ be a sub-class of $C([0,T]\times\mathbb{R})$ containing only bounded functions. $C_b([0,T]\times\mathbb{R})$ is a Banach space under the "sup"-norm

$$|u|_0 = \sup_{(t,x)\in[0,T]\times\mathbb{R}} |u(t,x)|.$$

In addition to the "vanilla" norm $|\cdot|_0$, we introduce a family of equivalent, weighted, norms $\{|\cdot|_{(\beta)} : \beta \geq 0\}$, given by

$$|u|_{(\beta)} = \sup_{(t,x)\in[0,T]\times\mathbb{R}} e^{-\beta(T-t)}|u(t,x)|, \text{ for } \beta\geq 0.$$

Due to the importance of one-sided bounds, we use the following notation

$$\lceil u\rceil = \sup_{(t,x)\in[0,T]\times\mathbb{R}} u(t,x),\ \lfloor u\rfloor = \inf_{(t,x)\in[0,T]\times\mathbb{R}} u(t,x),$$

as well as their section-wise counterparts

$$\lceil u(t,\cdot)\rceil = \sup_{x\in\mathbb{R}} u(t,x),\ \lfloor u(t,\cdot)\rfloor = \inf_{x\in\mathbb{R}} u(t,x).$$

The **parabolic distance** d_p between (t_1,x_1) and (t_2,x_2) in $[0,T]\times\mathbb{R}$ is defined by

$$d_p\Big((t_1,x_1),(t_2,x_2)\Big) = \sqrt{|t_1-t_2|} + |x_1-x_2|.$$

For a function $u \in C([0,T]\times\mathbb{R})$ and a constant $\alpha \in (0,1]$, we define its α-**Hölder constant** $[u]_\alpha \in [0,\infty]$ by

$$[u]_\alpha = \sup_{(t_1,x_1)\neq(t_2,x_2)\in[0,T]\times\mathbb{R}} \frac{|u(t_1,x_1)-u(t_2,x_2)|}{d_p\Big((t_1,x_1),(t_2,x_2)\Big)^\alpha}. \tag{A.1}$$

The functional $|\cdot|_\alpha$, given by

$$|u|_\alpha = |u|_0 + [u]_\alpha, \tag{A.2}$$

is a norm and it turns the class $C^\alpha([0,T]\times\mathbb{R})$ of all functions $u\in C_b([0,T]\times\mathbb{R})$ for which $[u]_\alpha<\infty$ into a Banach space.

For $k\in\mathbb{N}$, the space $C^k([0,T]\times\mathbb{R})$ contains all functions $u\in C([0,T]\times\mathbb{R})$ such that the partial derivatives $\frac{\partial^{m+n}}{\partial t^m\partial x^n}u$ exist and are continuous on $(0,T)\times\mathbb{R}$, for all $m,n\in\mathbb{N}_0$ such that $2m+n\le k$. For $k\in\mathbb{N}$ and $\alpha\in(0,1]$ we introduce the space $C^{k+\alpha}([0,T]\times\mathbb{R})$ by

$$\begin{aligned} C^{k+\alpha}([0,T]\times\mathbb{R}) = \{u\in C^k([0,T]\times\mathbb{R}) :\ & \tfrac{\partial^{m+n}}{\partial x^n\partial t^m}u \text{ admit extensions in } C^\alpha([0,T]\times\mathbb{R}) \\ & \text{for all } m,n\in\mathbb{N}_0 \text{ such that } 2m+n\le k\}. \end{aligned} \tag{A.3}$$

The norm $|\cdot|_{k+\alpha}$, given by

$$|u|_{k+\alpha} = \sum_{2m+n=k}[\tfrac{\partial^{m+n}}{\partial x^n\partial t^m}u]_\alpha + \sum_{2m+n\le k}|\tfrac{\partial^{m+n}}{\partial x^n\partial t^m}u|_0 \ \text{ for } u\in C^{k+\alpha}([0,T]\times\mathbb{R}),$$

turns $C^{k+\alpha}([0,T]\times\mathbb{R})$ into a Banach space. In particular, we shall have occasion to use the spaces $C^{1+\alpha}([0,T]\times\mathbb{R})$ and $C^{2+\alpha}([0,T]\times\mathbb{R})$ with norms

$$|u|_{1+\alpha} = [u_x]_\alpha + |u|_0 + |u_x|_0, \text{ and}$$
$$|u|_{2+\alpha} = [u_t]_\alpha + [u_{xx}]_\alpha + |u_t|_0 + |u_{xx}|_0 + |u_x|_0 + |u|_0.$$

Analogous constructions can be performed in the **isotropic** setting, i.e., in our case, for functions u of a single variable. For $\alpha\in(0,1]$, in an act of notation overload, we set

$$[u]_\alpha = \sup_{x_1\neq x_2\in\mathbb{R}} \frac{u(x_1)-u(x_2)}{|x_1-x_2|^\alpha},\ |u|_\alpha = [u]_\alpha + \sup_{x\in\mathbb{R}}|u(x)|.$$

Then, the Hölder space $C^{k+\alpha}(\mathbb{R})$, $k\in\mathbb{N}_0$, is a linear space consisting of all functions $u:\mathbb{R}\to\mathbb{R}$ which admit k continuous derivatives and whose k^{th} derivative $\frac{d^k}{dx^k}u$ satisfies $[\frac{d^k}{dx^k}u]_\alpha<\infty$. It becomes a Banach space when endowed with the norm $|\cdot|_{k+\alpha}$ defined in analogy with its anisotropic counterpart. Further information on Hölder spaces can be found in a variety of classical treatments of parabolic PDEs; for example, the reader might want to consult [21] for an unbounded-domain setting similar to ours, or [22] and [30] for a more thorough analysis of linear and quasilinear parabolic PDEs on bounded domains.

In addition to the classical function spaces, we shall have occasion to use functions which depend on an additional variable $n\in\{0,1\}$. For $\alpha\in(0,1]$, and $k\in\mathbb{N}_0$, the space of all such functions for which $k+\alpha-$ Hölder continuity is required on both $n=0-$ and $n=1-$ slices, will be denoted by $C^{k+\alpha}([0,T]\times\mathbb{R}\times\{0,1\})$ (or $C^{k+\alpha}(\mathbb{R}\times\{0,1\})$ in the isotropic case). The natural (and Banach) norm used in those spaces is the maximum of the Hölder norms of the n-slices:

$$|u|_{k+\alpha} = \max(|u(\cdot,0)|_{k+\alpha}, |u(\cdot,1)|_{k+\alpha}),$$
$$[u]_{k+\alpha} = \max([u(\cdot,0)]_{k+\alpha}, [u(\cdot,1)]_{k+\alpha}).$$

Similarly, for $u \in C([0,T] \times \mathbb{R} \times \{0,1\})$, we define

$$\lceil u \rceil = \max(\lceil u(\cdot,\cdot,0) \rceil, \lceil u(\cdot,\cdot,1) \rceil), \text{ and } \lfloor u \rfloor = \min(\lfloor u(\cdot,\cdot,0) \rfloor, \lfloor u(\cdot,\cdot,1) \rfloor).$$

A.2. **Some useful results on Hölder spaces.** When dealing with various constants in the statements and proofs of results below, we use the following convention:

Convention 2.

(1) *The variables μ, γ and $\alpha \in (0,1)$ are considered "global" and will not change throughout the paper. Any function of the global variables (and global variables only) if called a* **universal** *constant.*

(2) *The notation $a \preceq b$ means that there exists a universal constant $C > 0$ such that $a \leq Cb$. Such a constant may change from line to line.*

We start with several well-known interpolation results which we rephrase (and minimally adjust).

Theorem A.1 (**Parabolic interpolation - additive form** - [21], Theorem 8.8.1., p. 124.)**.** *There exists a universal constant $C > 0$, such that for any $\varepsilon > 0$, and $u \in C^{2+\alpha}([0,T] \times \mathbb{R})$ we have*

$$\begin{aligned} [u]_\alpha &\leq \varepsilon [u]_{2+\alpha} + C\varepsilon^{-\alpha/2} |u|_0, \\ [u_x]_\alpha &\leq \varepsilon [u]_{2+\alpha} + C\varepsilon^{-(1+\alpha)} |u|_0, \\ |u_x|_0 &\leq \varepsilon [u]_{2+\alpha} + C\varepsilon^{-1/(1+\alpha)} |u|_0. \end{aligned}$$

Corollary A.2. *There exists a universal constant $C > 0$ such that any function $u \in C^{2+\alpha}([0,T] \times \mathbb{R})$ which satisfies the inequality*

$$[u]_{2+\alpha} \leq D + E|u|_0 + F[u]_\alpha + G|u_x|_0 + H[u_x]_\alpha,$$

for some constants $D, E, F, G \geq 0$, also satisfies the inequality

$$[u]_{2+\alpha} \leq C\Big(D + \Big[E + F^{1+\alpha/2} + G^{1+\frac{1}{1+\alpha}} + H^{2+\alpha}\Big]|u|_0\Big). \tag{A.4}$$

Proof. By Theorem A.1, with the choice of $\varepsilon = \frac{1}{6F}$, we have $[u]_\alpha \leq \frac{1}{6F}[u]_{2+\alpha} + C(6F)^{\alpha/2}|u|_0$, so that

$$F[u]_\alpha \leq \tfrac{1}{6}[u]_{2+\alpha} + CF^{1+\alpha/2}|u|_0.$$

Similarly,

$$G|u_x|_0 \leq \tfrac{1}{6}[u]_{2+\alpha} + CG^{1+\frac{1}{1+\alpha}}|u|_0 \quad \text{and} \quad H[u_x]_\alpha \leq \tfrac{1}{6}[u]_{2+\alpha} + CH^{2+\alpha}|u|_0.$$

The estimate (A.4) now follows from

$$\begin{aligned} [u]_{2+\alpha} &\leq D + E|u|_0 + F[u]_\alpha + G|u_x|_\alpha + H[u_x]_\alpha \\ &\leq D + \tfrac{1}{2}[u]_{2+\alpha} + C\Big[E + F^{1+\alpha/2} + G^{1+\frac{1}{1+\alpha}} + H^{2+\alpha}\Big]|u|_0. \end{aligned}$$

$\square$

Theorem A.3 (**Parabolic interpolation - multiplicative form -** [21], Exercise 8.8.2., p. 125.)**.** *There exists a universal constant $C > 0$ such that for any $u \in C^{2+\alpha}([0,T] \times \mathbb{R})$ we have*

$$\begin{aligned} [u]_\alpha^{2+\alpha} &\leq C[u]_{2+\alpha}^\alpha |u|_0^2, \\ |u_x|_0^{2+\alpha} &\leq C[u]_{2+\alpha}|u|_0^{1+\alpha}, \\ [u_x]_\alpha^{2+\alpha} &\leq C[u]_{2+\alpha}^{1+\alpha}|u|_0. \end{aligned} \tag{A.5}$$

Theorem A.4 (**A Hölder estimate -** based on [21], Exercise 9.1.4, p. 139.)**.** *There exists a universal constant $C > 0$ such that for any $u \in C^{2+\alpha}([0,T] \times \mathbb{R})$*

$$[u]_{2+\alpha} \leq C([u_t + \tfrac{1}{2}u_{xx}]_\alpha + [u(T,\cdot)]_{2+\alpha}). \tag{A.6}$$

Corollary A.5 (**A Hölder estimate with a transport term).** *There exists a universal constant $C > 0$ such that for any $u \in C^{2+\alpha}([0,T] \times \mathbb{R})$ and $h \in C^\alpha([0,T] \times \mathbb{R})$ we have*

$$[u]_{2+\alpha} \leq C\Big([u_t + \tfrac{1}{2}u_{xx} + hu_x]_\alpha + [u(T,\cdot)]_{2+\alpha} + \big(1 + |h|_\alpha^{2+\alpha}\big)|u|_0\Big). \tag{A.7}$$

Proof. The Hölder estimate of Theorem A.4 implies that for $f = u_t + \frac{1}{2}u_{xx} + hu_x$ and $g = u(T,\cdot)$ we have

$$[u]_{2+\alpha} \preceq [g]_{2+\alpha} + [f - hu_x]_\alpha \preceq [g]_{2+\alpha} + [f]_\alpha + |h|_0[u_x]_\alpha + [h]_\alpha|u_x|_0,$$

and (A.7) follows from Corollary A.2. □

We finish this section with a preparatory result which states that composition with a C^2-function $x \mapsto e^{\gamma x}$ is a locally Lipschitz mapping on $C^\alpha([0,T] \times \mathbb{R})$. Even though it is quite likely that such a result is well-known, we were unable to locate a reference, and so, for the sake of completeness, a proof is provided.

Lemma A.6. *For $\gamma \geq 0$, let $E_\gamma : C^\alpha([0,T] \times \mathbb{R}) \to C([0,T] \times \mathbb{R})$ be the composition mapping*

$$E_\gamma u(t,x) = e^{\gamma u(t,x)} \text{ for } u \in C^\alpha([0,T] \times \mathbb{R}).$$

Then $E_\gamma u \in C^\alpha([0,T] \times \mathbb{R})$ and the following bounds hold for all $u, u^{(1)}, u^{(2)} \in C^\alpha([0,T] \times \mathbb{R})$,

$$|E_\gamma u|_0 \leq e^{\gamma \lceil u \rceil}, \tag{A.8}$$

$$[E_\gamma u]_\alpha \leq \gamma e^{\gamma \lceil u \rceil}[u]_\alpha, \tag{A.9}$$

$$|E_\gamma u^{(2)} - E_\gamma u^{(1)}|_0 \leq \gamma D|u^{(2)} - u^{(1)}|_0, \text{ and} \tag{A.10}$$

$$[E_\gamma u^{(2)} - E_\gamma u^{(1)}]_\alpha \leq \gamma D\Big([u^{(2)} - u^{(1)}]_\alpha + \gamma[u^{(2)}]_\alpha|u^{(2)} - u^{(1)}|_0\Big), \tag{A.11}$$

where $D = e^{\gamma \lceil u^{(1)} \rceil \vee \lceil u^{(2)} \rceil}$.

Proof. Since $0 \leq E_\gamma u \leq \sup_{t,x} e^{\gamma u(t,x)}$, (A.8) holds. For (A.9), we note that, by the intermediate value theorem,

$$\left| e^{\gamma u(s,y)} - e^{\gamma u(t,x)} \right| = \gamma e^{\gamma \xi} \left| u(s,y) - u(t,x) \right|, \tag{A.12}$$

for some convex combination ξ of $u(t,x)$ and $u(s,y)$, so that $e^{\gamma\xi} \leq e^{\gamma \lceil u \rceil}$.

In order to get the other two bounds, we pick $u^{(1)}, u^{(2)} \in C^{\alpha}([0,T]\times\mathbb{R})$, (t_1,x_1), $(t_2,x_2)\in[0,T]\times\mathbb{R}$ and set $d = d_p\Big((t_1,x_1),(t_2,x_2)\Big)$ and $\delta u^{(k)} = u^{(k)}(t_2,x_2) - u^{(k)}(t_1,x_1)$ for $k=1,2$. Observe that we can prove (A.10) in a similar manner as (A.9). Focusing on (A.11), we set

$$\delta = e^{\gamma u^{(2)}(t_2,x_2)} - e^{\gamma u^{(1)}(t_2,x_2)} - (e^{\gamma u^{(2)}(t_1,x_1)} - e^{\gamma u^{(1)}(t_1,x_1)})$$

and note the elementary equality

$$G(b) - G(a) = (b-a)\int_0^1 G'(h^\theta(a,b))\,d\theta,$$

where $h^\theta(a,b) = (1-\theta)a + \theta b$, $G\in C^1(\mathbb{R})$. Then,

$$\begin{aligned}|\delta| &= \Big|\delta u^{(2)}\int_0^1 \gamma e^{\gamma h^\theta(u^{(2)}(t_1,x_1),u^{(2)}(t_2,x_2))}\,d\theta - \delta u^{(1)}\int_0^1 \gamma e^{\gamma h^\theta(u^{(1)}(t_1,x_1),u^{(1)}(t_2,x_2))}\,d\theta\Big|\\ &\le \gamma D\Big|\delta u^{(2)} - \delta u^{(1)}\Big| + \Big|\delta u^{(2)}\int_0^1 \delta h(\theta)\int_0^1 \gamma^2 e^{\gamma g(\eta,\theta)}\,d\eta\,d\theta\Big|,\end{aligned}$$

where

$$\begin{aligned}\delta h(\theta) &= h^\theta(u^{(2)}(t_1,x_1) - u^{(1)}(t_1,x_1), u^{(2)}(t_2,x_2) - u^{(1)}(t_2,x_2)),\\ g(\eta,\theta) &= h^\eta(h^\theta(u^{(1)}(t_1,x_1),u^{(1)}(t_2,x_2)), h^\theta(u^{(2)}(t_1,x_1),u^{(2)}(t_2,x_2))).\end{aligned}$$

Clearly, $g(\eta,\theta)$ is a convex combination of values of functions $u^{(1)}$ and $u^{(2)}$ and the inequality $\delta h(\theta) \le |u^{(2)} - u^{(1)}|_0$ holds. Therefore,

$$|\delta| \le D\Big([u^{(2)} - u^{(1)}]_\alpha \gamma d^\alpha + [u^{(2)}]_\alpha |u^{(2)} - u^{(1)}|_0 \gamma^2 d^\alpha\Big), \tag{A.13}$$

which directly implies (A.11). □

REFERENCES

[1] R. M. ANDERSON AND R. C. RAIMONDO, "Equilibrium in continuous-time financial markets: Endogenously dynamically complete markets", *Econometrica*, vol. 76, no. 4, pp. 841–907, 2008.

[2] S. BASAK AND D. CUOCO, "An equilibrium model with restricted stock market participation", *Review of Financial Studies*, vol. 11, no. 2, pp. 309–41, 1998.

[3] D. BECHERER, *Rational Hedging and Valuation with Utility-based Preferences.* PhD thesis, Technical University of Berlin, 2001.

[4] L. CARASSUS AND M. RÁSONYI, "Convergence of utility indifference prices to the superreplication price", *Math. Methods Oper. Res.*, vol. 64, no. 1, pp. 145–154, 2006.

[5] ——, "Optimal strategies and utility-based prices converge when agents' preferences do", *Math. Oper. Res.*, vol. 32, no. 1, pp. 102–117, 2007.

[6] R. Carmona, ed., *Indifference pricing.* Princeton Series in Financial Engineering, Princeton, NJ: Princeton University Press, 2009. Theory and applications.

[7] R.-A. DANA AND M. PONTIER, "On existence of an Arrow-Radner equilibrium in the case of complete markets. A remark", *Math. Oper. Res.*, vol. 17, no. 1, pp. 148–163, 1992.

[8] F. DELBAEN, P. GRANDITS, T. RHEINLÄNDER, D. SAMPERI, M. SCHWEIZER, AND C. STRICKER, "Exponential hedging and entropic penalties", *Math. Finance*, vol. 12, no. 2, pp. 99–123, 2002.

[9] D. DUFFIE, "Stochastic equilibria: existence, spanning number, and the "no expected financial gain from trade" hypothesis", *Econometrica*, vol. 54, no. 5, pp. 1161–1183, 1986.

[10] D. DUFFIE AND C. F. HUANG, "Implementing Arrow-Debreu equilibria by continuous trading of few long-lived securities", *Econometrica*, vol. 53, no. 6, pp. 1337–1356, 1985.

[11] D. DUFFIE AND W. ZAME, "The consumption-based capital asset pricing model", *Econometrica*, vol. 57, pp. 1279–1297, 1989.

[12] W. H. Fleming and H. M. Soner, *Controlled Markov processes and viscosity solutions*, vol. 25 of *Applications of Mathematics (New York)*. New York: Springer-Verlag, 1993.

[13] J. Hadamard, "Sur les problèmes aux dérivées partielles et leur signification physique", *Princeton University Bulletin*, pp. 49–52, 1902.

[14] J. Hugonnier, "Bubbles and multiplicity of equilibria under portfolio constraints", preprint, 2008.

[15] E. Jouini and C. Napp, "Convergence of utility functions and convergence of optimal strategies", *Finance Stoch.*, vol. 8, no. 1, pp. 133–144, 2004.

[16] I. Karatzas, P. Lakner, J. P. Lehoczky, and S. E. Shreve, "Equilibrium in a simplified dynamic, stochastic economy with heterogeneous agents", in *Stochastic analysis*, pp. 245–272, Boston, MA: Academic Press, 1991.

[17] I. Karatzas, J. P. Lehoczky, and S. E. Shreve, "Existence and uniqueness of multi-agent equilibrium in a stochastic, dynamic consumption/investment model", *Math. Oper. Res.*, vol. 15, no. 1, pp. 80–128, 1990.

[18] ———, "Equilibrium models with singular asset prices", *Math. Finance*, vol. 1, pp. 11–29, 1991.

[19] I. Karatzas and S. E. Shreve, *Methods of mathematical finance*, vol. 39 of *Applications of Mathematics (New York)*. New York: Springer-Verlag, 1998.

[20] C. Kardaras and G. Žitković, "Stability of the utility maximization problem with random endowment in incomplete markets", to appear in Mathematical Finance, 2007.

[21] N. V. Krylov, *Lectures on elliptic and parabolic equations in Hölder spaces*, vol. 12 of *Graduate Studies in Mathematics*. Providence, RI: American Mathematical Society, 1996.

[22] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural′ceva, *Linear and quasilinear equations of parabolic type*. Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23, Providence, R.I.: American Mathematical Society, 1967.

[23] K. Larsen, "Continuity of utility-maximization with respect to preferences", to appear in Mathematical Finance, 2007.

[24] K. Larsen and G. Žitković, "Stability of utility-maximization in incomplete markets", *Stochastic Processes and their Applications*, vol. 117, no. 11, pp. 1642–1662, 2007.

[25] M. Magill and M. Quinzii, *Theory of Incomplete Markets, Volume 1*. Cambridge and London: MIT Press, 1996.

[26] M. Owen and G. Žitković, "Optimal investment with an unbounded random endowment and utility-based pricing", to appear in Mathematical Finance, 2006.

[27] D. Revuz and M. Yor, *Continuous martingales and Brownian motion*, vol. 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Berlin: Springer-Verlag, third ed., 1999.

[28] H. Scarf, "The approximation of fixed points of a continuous mapping", *SIAM J. Appl. Math.*, vol. 15, pp. 1328–1343, 1967.

[29] H. E. Scarf, "On the computation of equilibrium prices", Cowles Foundation Discussion Papers 232, Cowles Foundation, Yale University, 1967.

[30] Z. Wu, J. Yin, and C. Wang, *Elliptic & parabolic equations*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2006.

[31] G. Žitković, "Financial equilibria in the semimartingale setting: complete markets and markets with withdrawal constraints", *Finance and Stochastics*, vol. 10, no. 1, pp. 99–119, 2006.